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The Effect of a "Bump" in the Turbulence Spectrum on Laser Propagation

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1 Introduction

This report will address the effect of a "bump" in the turbulence spectrum on the propagation of a laser beam. The motivation for this work is as follows: In conventional treatments on the effect of turbulence on laser propagation, it is usually assumed that the turbulence can be described by a spectral density which represents a *cascade* of energy from long wavelength modes down to shorter wavelength modes. The most common assumption is that of the *Kolmogorov model*, where in the "inertial" range of the spectrum (between a minimum wavenumber k_{L_0} and maximum wavenumber k_{l_0}, corresponding to the maximum size L_0 and minimum size l_0 of the turbulent eddies) the spectral density decreases as $k^{-(3d+2)/3}$ (in d dimensions). In some media however, a source for injecting energy into the turbulent spectrum at a particular discrete wavelength (or over a range of wavelengths) may be present: for example, this is the case in a turbulent plasma where an instability may be growing over a range of wavelengths. The presence of such an energy source will thus produce a "bump" in the turbulence spectrum, as shown schematically in Fig.1. As we shall see below, the spectrum can then be viewed as this "bump" superimposed upon the background natural cascade: that is, the spectrum can be analyzed as the sum of a cascade spectrum and the "bump" spectrum, and the effects of each on the propagation of the beam can thus be analyzed separately.

This report is organized as follows. In Section 2 we will briefly review the mathematical formalism that will be used to analyze the statistical features of the effect of turbulence on laser propagation. In Section 3, this formalism will be applied to the case where the turbulence is described by a spectral density which is a "bump" at some finite wave number k_b. Section 3.2 will treat the idealized (but analytically tractable) case where the "bump" is a δ-function; the effect of a "bump" of finite width is studied numerically in Section 3.3. Finally, the superposition of a "bump" on a cascade-type spectrum is considered in Section 3.4. One important conclusion which results from this study is that in three dimensions, even if the amplitude of the "bump" component of the spectrum is much smaller than the amplitude of the cascade component (for example, by ten orders of magnitude), if the scalelength of the turbulence represented by the "bump" is much smaller than the scalelength of the cascade turbulence (say four orders of magnitude smaller), then *almost all* of the power in the spectrum is due to the "bump" component. Furthermore, in this case, the effects of the turbulence on the beam are in effect determined by the "bump" component, and the existence of the cascade component (which may represent turbulence at very large scalelengths) can be ignored.

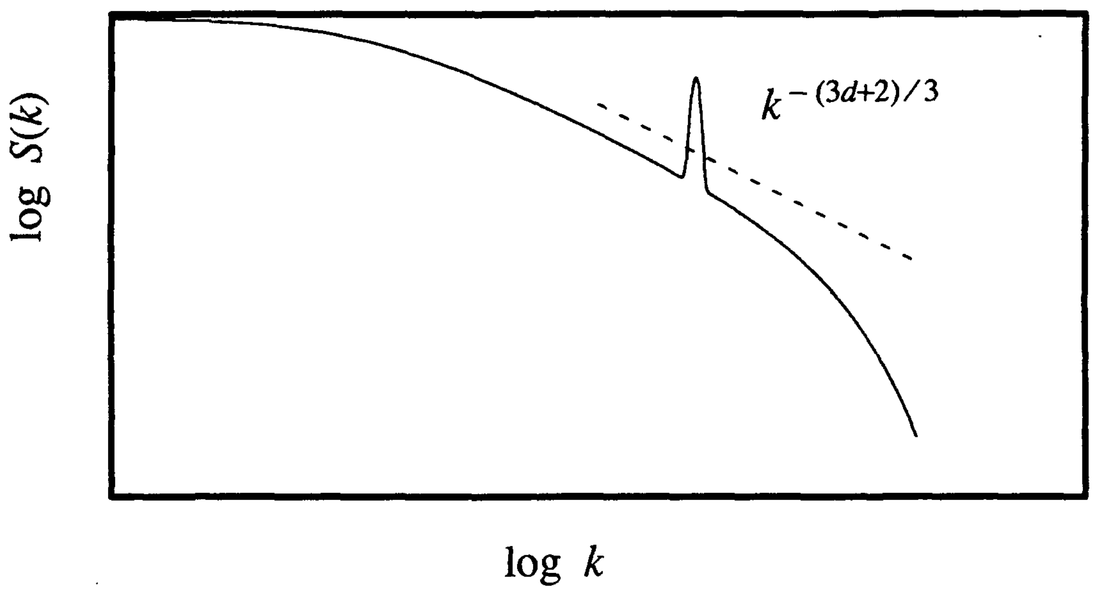

Figure 1. Example of a turbulence spectrum with a "bump" superimposed on a cascade.

2 Preliminaries

In this Section we will briefly review the mathematical constructions required to analyze the statistical effects of a turbulent medium on the propagation of a laser beam. For simplicity, we will assume that the problem is two-dimensional: the beam (with wavenumber k_0) is taken to be propagating in the z-direction, with a transverse beam profile $\psi(x)$ on the x-axis. The extension of mathematical expressions to three dimensions is straightforward, and the main results are not changed. Furthermore, we shall assume throughout this article that the laser wavelength $\lambda \equiv 2\pi/k_0$ is much shorter than any scalelength of variation of the medium (*i.e.*, the minimum size l_0 of the turbulent eddies) so that we are working in the eikonal (or WKB) regime.

If the disturbance of the beam due to the turbulent medium is small over a propagation distance z, the beam can be thought of as a bundle of rays propagating roughly parallel to the z-direction. The phase shift ϕ along a ray propagating at the transverse location x caused by the turbulent perturbations $\delta N(x,z)$ in the media's index of refraction is therefore

$$\phi(x) = k_0 \int_0^z dz' \, \delta N(x,z') \tag{1}$$

In this expression, ϕ is the phase shift induced by a single realization of the turbulence, characterized by δN.

2.1 Turbulence and Phase Correlation Functions

A statistical analysis of the effect of the turbulence on beam propagation is based on averages of characteristic quantities (such as this phase shift) over an ensemble of realizations of the turbulence. Denoting this average by the angle brackets $\langle\cdot\rangle$, we assume that the turbulent fluctuations have zero mean, $\langle \delta N(x,z)\rangle = 0$, so that the mean phase shift from (1) also vanishes. The correlation function of the phase is defined as

$$\begin{aligned} R_\phi(|x-x'|;z) &\equiv \langle \phi(x)\phi(x')\rangle = k_0^2 \int_0^z dz'\,dz'' \,\langle \delta N(x,z')\delta N(x',z'')\rangle \\ &= k_0^2 \int_0^z dz'\,dz'' \, R_N\left(\sqrt{(x'-x)^2+(z''-z')^2}\right) \end{aligned} \tag{2}$$

Here we have assumed that the turbulence is isotropic and homogeneous: the refractive index correlation function R_N depends only the the spatial separation of the two points (x,z') and (x',z''). In this case, the phase correlation function depends only on the separation $|x-x'|$ on the transverse x-axis (and the propagation distance z). Because of this assumption, this expression can be simplified: changing variables $s_x = x'-x$, $s_z = z''-z'$, it can be recast as

$$R_\phi(|s_x|;z) = k_0^2 \int_0^z dz' \int_{-z'}^{z-z'} ds_z \; R_N\left(\sqrt{s_x^2+s_z^2}\right) \tag{3}$$

The area in (z', s_z)-space over which the integral is to be performed is shown in Fig.2a: here, the integral in the s_z-direction is to be performed with z' fixed, and then z' is varied. Changing this order of integration requires changing the limits of integration: holding s_z fixed, first integrate z' from $-s_z$ to z (for $s_z < 0$), or from 0 to $z - s_z$ (for $s_z > 0$), and then vary s_z from $-z$ to z (see Fig.2b).

$$\begin{aligned} R_\phi(|s_x|\,; z) &= k_0^2 \int_{-z}^{0} ds_z \int_{-s_z}^{z} dz'\, R_N\left(\sqrt{s_x^2 + s_z^2}\right) \\ &\quad + k_0^2 \int_0^z ds_z \int_0^{z-s_z} dz'\, R_N\left(\sqrt{s_x^2 + s_z^2}\right) \end{aligned} \tag{4}$$

The integrals over z' can now be performed; then letting $s_z \to -s_z$ in the first integral, the two integrals can be combined to give

$$R_\phi(|s_x|\,; z) = 2k_0^2 \int_0^z ds_z\, (z - s_z)\, R_N\left(\sqrt{s_x^2 + s_z^2}\right) \tag{5}$$

We note that this is an exact expression for the relationship between the correlation function R_ϕ of the wave phase and the correlation function R_N of the refractive index fluctuations, based only on the definition (1) for the phase shift. In practice, however, R_N typically decays to zero over some correlation distance L_0 (often called the outer scale of the turbulence), and this distance is much less than the propagation distance of interest ($L_0 \ll z$). In this case, the maximum value of s_z that contributes to the integral (for fixed s_x) is about L_0, so that $z - s_z \approx z$ is a very good approximation. Taking this term outside the integral then, we have

$$R_\phi(|s_x|\,; z) \approx 2k_0^2 z \int_0^\infty ds_z\, R_N\left(\sqrt{s_x^2 + s_z^2}\right) \tag{6}$$

where the same approximation allows us to extend the limit of integration to infinity.

2.2 Turbulence and Phase Spectral Densities

The spectral density of a homogeneous, isotropic random process (such as the turbulent index of refraction or the induced phase shift of the laser beam) is defined as the Fourier transform of the correlation function. For the two-dimensional turbulence we have

$$\begin{aligned} S_N(k_x, k_z) &\equiv \int ds_x\, ds_z\, R_N\left(\sqrt{s_x^2 + s_z^2}\right)\, e^{-ik_x s_x} e^{-ik_z s_z} \\ &= \int s\, ds\, R_N(s) \int_0^{2\pi} d\theta\, e^{-iks\cos\theta} \\ S_N(k) &= 2\pi \int s\, ds\, R_N(s)\, J_0(ks) \end{aligned} \tag{7}$$

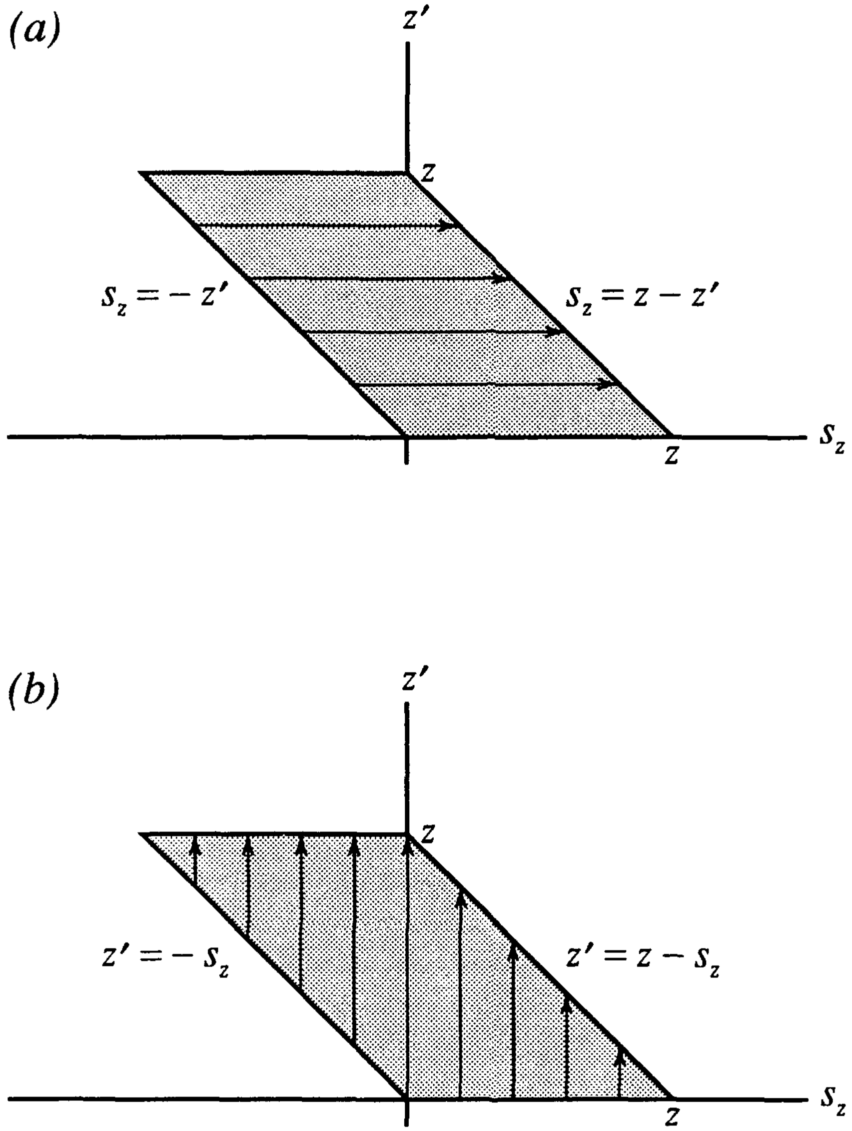

Figure 2. Area over which (z', s_z) integrals to be performed in (3). (a) Original order of integration. (b). Inverted order of integration.

Since the turbulence is homogeneous and isotropic in physical space, it is isotropic in k-space as well. The inverse of (7) is similar:

$$\begin{aligned} R_N(s_x, s_z) &\equiv \int \frac{dk_x}{2\pi}\frac{dk_z}{2\pi} S_N\left(\sqrt{k_x^2 + k_z^2}\right) e^{ik_x s_x} e^{ik_z s_z} \\ &= \int \frac{k\,dk}{2\pi} S_N(k) \int_0^{2\pi} \frac{d\theta}{2\pi} e^{iks\cos\theta} \\ R_N(s) &= \int \frac{k\,dk}{2\pi} S_N(k)\, J_0(ks) \end{aligned} \tag{8}$$

In practice, the spectral density is often the more fundamental quantity in the sense that a turbulence model is typically specified by postulating the form of the spectral density S_N. One reason for this is that it can be shown that the spectral density of a physically realizable random process must be non-negative. Thus, specifying a model for the correlation function R_N may produce by construction (7) a spectral density which does not satisfy this criterion; beginning with a form for S_N avoids this problem, and the correlation function can then be constructed from (8).

The spectral density associated with the one-dimensional phase correlation function is defined by

$$\begin{aligned} S_\phi(k_x) &\equiv \int ds_x\, R_\phi(|s_x|)\, e^{-ik_x s_x} \\ R_\phi(s_x) &= \int \frac{dk_x}{2\pi} S_\phi(k_x)\, e^{ik_x s_x} \end{aligned} \tag{9}$$

Note that since the phase fluctuations are a one-dimensional random process (in this two-dimensional propagation model), the Fourier transforms in (9) are one-dimensional (as opposed to those in (7,8) for the two-dimensional random process of the refractive index fluctuations).

Since the phase correlation function and refractive index correlation function are related by (5), a relation between their associated spectral densities can also be found. This is most easily achieved by introducing the first line of (8) into (2) to obtain

$$\begin{aligned} R_\phi(s_x) &= k_0^2 \int_0^z dz'\, dz''\, R_N\left(\sqrt{s_x^2 + (z'' - z')^2}\right) \\ &= k_0^2 \int_0^z dz'\, dz'' \int \frac{dk_x}{2\pi}\frac{dk_z}{2\pi} S_N(k)\, e^{ik_x s_x} e^{ik_z(z''-z')} \end{aligned} \tag{10}$$

The integrals over z' and z'' can be performed giving

$$R_\phi(s_x) = k_0^2 z^2 \int \frac{dk_x}{2\pi}\frac{dk_z}{2\pi} S_N(k)\, e^{ik_x s_x} \left(\frac{\sin k_z z/2}{k_z z/2}\right)^2 \tag{11}$$

Finally, performing the Fourier transform of both sides to obtain $S_\phi(k_x)$ (as in (9)) the result is

$$S_\phi(k_x) = k_0^2 z^2 \int \frac{dk_z}{2\pi}\; S_N(k) \left(\frac{\sin k_z z/2}{k_z z/2}\right)^2 \tag{12}$$

This expression could be obtained beginning with (5) as well.

The interpretation of (12) is that the spectral density of the phase fluctuations on the transverse x-axis is just the spectral density of the two-dimensional refractive index fluctuations *projected* onto the x-axis, but *filtered* so that only turbulent perturbations with wavelengths in the direction of propagation (the z-direction) comparable to the propagation distance z contribute. Two observations are immediate. First, if the spectral density of the turbulence S_N is non-negative (as it must be to describe a physically realizable random process), then (12) clearly shows that the phase spectrum S_ϕ is non-negative (note that this relation follows exactly from the premise (1) for the wave phase shift). This fact is extremely important in practice, where the so-called *phase-screen* method is often used to numerically propagate the laser beam through a turbulent medium. In that method, the wave is propagated as though in a uniform non-turbulent medium a distance z between "phase-shifting screens" which represent the effect of the turbulence; on each "screen", the wave $\psi(x)$ is modulated by a factor $\exp(i\phi(x;z))$, where $\phi(x;z)$ is a random phase function. The phase function $\phi(x;z)$ is generated by first constructing its Fourier transform $\hat{\phi}(k_x;z)$ from a Gaussian random distribution with variance $S_\phi(k_x)$ at each k_x, and then transforming this into x-space. It is straightforward to show that this method will produce the correct phase statistics (*i.e.*, the specified spectral density and correlation function of the phase) over an ensemble of a large number of phase screens. This method therefore inherently requires that $S_\phi(k_x)$ be non-negative, since its value at each k_x is used as the variance of a probability distribution. As we have seen above, this quality is guaranteed by first specifying the spectral density S_N of the refractive index fluctuations to be non-negative, and then constructing S_ϕ from (12).

The integral in (12) presents no particular problem for numerical computation, and in fact should be quite easy to evaluate in most cases of interest since the "filter function" $[\sin k_z z/2/(k_z z/2)]^2$ falls off rapidly (its first zero is at $k_z = 2\pi/z$). Our second observation, however, is that if S_N is slowly varying in the vicinity of $k_z = 0$ (for all k_x) then (12) can be approximated by setting $k_z = 0$ in the argument of S_N and performing the integral only over the "filter function" to obtain

$$S_\phi(k_x) = k_0^2 z\; S_N(k_x, k_z = 0) \tag{13}$$

Again we see that $S_\phi \geq 0$ if S_N is. The only requirement for (13) to hold is that S_N varies slowly in the k_z-direction in the vicinity $k_z z/2 < \pi$ for all values of k_x; thus, only wavelengths in the direction of propagation comparable to z are admitted,

and their contribution is approximated by a flat distribution at the $k_z = 0$ value. It is furthermore quite simple to show that while the inverse Fourier transform of the exact relation (12) yields (5) (as expected), the inverse transform of the approximate relation (13) produces the approximation (6); thus, the approximations leading to both (6) and (13) from (5) and (12), respectively, are consistent.

As a final note, one can relate the level of the phase fluctuations to the level of the refractive index fluctuations. These quantities (which have been assumed to be independent of location) are defined as

$$\langle \delta N^2 \rangle \equiv R_N(0) \tag{14}$$

$$\langle \phi^2 \rangle \equiv R_\phi(0) \tag{15}$$

Using (8) and (9), the value of either correlation function R_N or R_ϕ at the origin is given by an integral over the associated spectral density. Thus, using the approximate form (13) we have

$$\begin{aligned} \langle \phi^2 \rangle &= R_\phi(0) = \int \frac{dk_x}{2\pi} \, S_\phi(k_x) \\ &= k_0^2 z \int \frac{dk_x}{2\pi} S_N(k_x, k_z = 0) \\ &= k_0^2 z \, \langle \delta N^2 \rangle \, \ell_i \end{aligned} \tag{16}$$

The one-dimensional integral over S_N defines the *integral scalelength* ℓ_i of the turbulence.

2.3 Average Intensity

We conclude this Section with a discussion of the effect of a random medium (characterized by a spectral density for the refractive index fluctuations) on the intensity profile of a laser beam. In the absence of turbulence (or other effects of the medium, such as thermal blooming), a coherent beam of initial radius r_0 will spread due to natural diffraction according to

$$\frac{r^2(z)}{r_0^2} = 1 + \frac{1}{4}\left(\frac{z}{L_R}\right)^2 \tag{17}$$

where $L_R \equiv k_0 r_0^2$ is called the Rayleigh length of the beam.* This spreading can be viewed as the coherent interference of a bundle of rays propagating within an angle

*This expression is for two-dimensional propagation (in three dimensions, the factor of 1/4 becomes unity). In this way, L_R is the same in both two and three dimensions, for a given initial radius (or half-width) r_0 (defined by (26) in both cases); the difference then is that in two dimensions the width will double in a distance $z = \sqrt{12} L_R$, while in three dimensions the area will double at $z = L_R$.

$\theta = k_\perp/k_0 \sim (k_0 r_0)^{-1}$ of the z-axis, where the transverse wavenumber $k_\perp$ is inversely proportional to the initial beam radius. In a turbulent medium, however, we see from (1) that the phase along each ray develops a random component depending on its path through the randomly fluctuating index of refraction. Thus, if the level of the refractive index fluctuations is large enough, or the propagation distance long enough (so that the phase fluctuations are large enough), the effects of these random phase shifts in the the interference of the rays will begin to be seen. Two major effects are a loss of transverse beam coherence, and beam broadening due to the turbulence. The loss of coherence is manifested by a loss of smoothness in the transverse intensity profile; the intensity develops "spikes" so that it appears to have broken up into smaller-diameter "beamlets". Beam broadening can be thought of either as the result of a non-coherent, mostly forward scattering of the rays off of the turbulent refractive index fluctuations, or as a result of the more rapid natural broadening of the smaller-diameter beamlets (with smaller Rayleigh lengths) produced by the loss of coherence.

Due to the randomness in the turbulence, it is impossible to predict the evolution of the laser intensity profile as it propagates through a single realization of the turbulence. However, over many realizations of the turbulence governed by a spectral density S_N (or, over many laser shots through an ever-fluctuating medium), predictions can be made regarding the average behavior of the beam intensity and its characteristics (such as its width or coherence length). For example, a prediction for the ensemble-averaged wave intensity $\langle I(x,z)\rangle \equiv \langle|\psi(x,z)|^2\rangle$ can be derived[1] based on a modified Huygens principle:

$$\langle I(x,z)\rangle = \frac{k_0}{2\pi z}\int ds\; M_T(s/2;z)\, M_A(s;z)\, e^{-ik_0 xs/z} \tag{18}$$

Here, the *turbulence modulation correlation function* $M_T(s;z)$ is defined as

$$M_T(s;z) \equiv \langle e^{i\phi(x;z)}\, e^{-i\phi(x';z)}\rangle \tag{19}$$

while the *amplitude correlation* $M_A(s;z)$ is

$$M_A(s;z) \equiv \int dx\; \psi(x+\tfrac{1}{2}s,0)\,\psi(x-\tfrac{1}{2}s,0)\, e^{ik_0 xs/z} \tag{20}$$

It can be shown[1] that if the random phase shift $\phi(x,z)$ is a Gaussian random variable at each x, then the turbulence modulation correlation function becomes

$$M_T(s;z) = e^{-\langle\phi^2\rangle}\, e^{R_\phi(s)} \tag{21}$$

where $\langle\phi^2\rangle$ is the level of the phase fluctuations (16) and $R_\phi(s)$ is the phase correlation function. We note that since $M_T(0;z) = 1$, the total power in the beam (*i.e.*, the integral of (18) over x) remains constant as it propagates in z.

For an initial Gaussian amplitude profile

$$\psi(x,0) = e^{-x^2/2\sigma^2} \tag{22}$$

the amplitude function $M_A(s;z)$ is

$$M_A(s;z) = \sqrt{\pi}\sigma\, e^{-s^2 k_0^2 \sigma^2(z)/4z^2} \tag{23}$$

where

$$\sigma^2(z) \equiv \sigma^2 \left(1 + \frac{z^2}{k_0^2 \sigma^4}\right) \tag{24}$$

Note that in the absence of turbulence ($\langle \phi^2 \rangle = 0$), $M_T(s;z)$ is unity so that (23) in (18) yields the correct expression for the propagation of a Gaussian pulse in vacuum:

$$I(x,z) = \frac{\sigma}{\sigma(z)}\, e^{-x^2/\sigma^2(z)} \tag{25}$$

Furthermore, (24) clearly shows the effect of natural diffraction as stated in (17), with $r(z) = \sigma(z)/\sqrt{2}$.

One use of the expression (18) is to compute the degree of beam broadening due to turbulence. Defining the width r of the beam to be

$$r^2(z) \equiv \frac{\int dx\ x^2\ \langle I(x;z)\rangle}{\int dx\ \langle I(x;z)\rangle} \tag{26}$$

one can show[1] that the width grows as[†]

$$\frac{r^2(z)}{r_0^2} = 1 + \frac{1}{4}\left(\frac{z}{L_R}\right)^2 + \left(\frac{z}{L_T}\right)^3 \tag{27}$$

Here, the *turbulent broadening length* L_T is defined as

$$L_T \equiv \left[\frac{r_0^2 \ell_t^2}{\pi^2 \ell_i \langle \delta N^2 \rangle}\right]^{1/3} \tag{28}$$

in terms of the level of refractive index fluctuations $\langle \delta N^2 \rangle$, the integral scalelength ℓ_i of the turbulence (16), and the *transverse scalelength* of the turbulence ℓ_t:

$$\left(\frac{2\pi}{\ell_t}\right)^2 \equiv \frac{\int dk\ k^2 S_N(k)}{\int dk\ S_N(k)} \tag{29}$$

[†]As in (17), the factor of 1/4 becomes unity in three dimensions, and the length L_T is the same in both two and three dimensions for a given radius (or half-width) r_0.

Comparing (27) with (17), we see that if the turbulent broadening length is comparable to or less than the Rayleigh length, then turbulent broadening will dominate that due to natural diffraction, and the beam will broaden much more rapidly with the propagation distance.

The relation (18) describes the mean evolution of the laser intensity (averaged over many laser shots through different realizations of the turbulence). An expression can also be derived[1] for the mean evolution of the wave spectrum, $\langle I(k,z)\rangle \equiv \langle \left|\hat{\psi}(k,z)\right|^2 \rangle$:

$$\langle I(k,z)\rangle = \int \frac{d\kappa}{2\pi} \left|\hat{\psi}(\kappa,0)\right|^2 \hat{M}_T(k-\kappa;z) \tag{30}$$

where $\hat{\psi}(k,z)$ is the Fourier transform of $\psi(x,z)$ and $\hat{M}_T$ is the Fourier transform of M_T

$$\hat{M}_T(k;z) \equiv \int ds\; M_T(s;z)\, e^{-iks} \tag{31}$$

In the absence of turbulence, $M_T(s;z) = 1$ so that $\hat{M}_T(k;z) = 2\pi\delta(k)$; thus we see that, as expected, the wave spectrum from (30) does not change as the wave propagates in vacuum. Scattering of the wave by turbulent fluctuations will produce changes in the spectrum, primarily by causing a shift to larger and larger transverse wavenumbers; this then is the mechanism for a broadening of the wave spectrum due to turbulence.

3 The Effect of a "Bump" in the Spectrum

In this Section we will use the mathematical formalism of the preceding Section to analyze the effect of a "bump" in the turbulence spectrum on the propagation of a laser beam.

3.1 Discussion of Superposition

We first assume that we are given an isotropic two-dimensional turbulence spectrum $S_N(k)$ (depending only on the magnitude of $k = \sqrt{k_x^2 + k_z^2}$) similar to that shown in Fig.1. Such a spectrum can be assumed to be a superposition of a "bump" spectrum $S_N^{(b)}$ on a natural cascade spectrum $S_N^{(c)}$, or

$$S_N(k) \equiv S_N^{(c)}(k) + S_N^{(b)}(k) \tag{32}$$

Since all the expressions involving the spectral density in the preceding Section (such as the Fourier transforms in (7–9) and the relations between the phase and turbulence spectra (12,13)) are linear in the spectral density, it would appear that we could treat each component of the spectrum ($S_N^{(c)}$ and $S_N^{(b)}$) independently and then linearly superpose the results.

Consider, however, what this means in terms of the numerical phase-screen method. As briefly described in Section 2.2, this method is based on first constructing a random phase shift $\hat{\phi}(k)$ at each value of k from a Gaussian random probability distribution with the value of $S_\phi(k)$ as the variance. The justification for this comes from the following considerations. From (9) we have (with $k_x \rightarrow k$)

$$R_\phi(|x - x'|) = \langle \phi(x)\phi(x') \rangle = \int \frac{dk}{2\pi}\, S_\phi(k)\, e^{ik(x-x')} \tag{33}$$

Now performing the Fourier transform over x and x' we obtain

$$\begin{aligned} \langle \hat{\phi}(k')\hat{\phi}^*(k'') \rangle &= \int dx\, dx'\, e^{-ik'x}\, e^{ik''x'} \int \frac{dk}{2\pi}\, S_\phi(k)\, e^{ik(x-x')} \\ &= 2\pi\, S_\phi(k')\, \delta(k' - k'') \end{aligned} \tag{34}$$

This shows that a homogeneous random process (such as the phase $\phi(x)$) is characterized by a Fourier transform $\hat{\phi}(k)$ which is an independent random variable with variance $S_\phi(k)$ at each k.

Now suppose we have the case as in (32) where the spectrum S_ϕ (constructed from S_N by either of the linear relations (12) or (13)) is

$$S_\phi(k) \equiv S_\phi^{(c)}(k) + S_\phi^{(b)}(k) \tag{35}$$

To consider the effects of each component of the spectrum separately and then superpose the results means that we should be able to separately construct a phase shift $\phi_c(x)$ (from $\hat{\phi}_c(k)$ using $S_\phi^{(c)}(k)$ as a variance) and a phase shift $\phi_b(x)$ (from $\hat{\phi}_b(k)$ using $S_\phi^{(b)}(k)$ as a variance) and then add them together for a total phase shift $\phi(x) = \phi_c(x) + \phi_b(x)$ which has the statistics (over an ensemble of phase screen realizations) of the total spectrum S_ϕ. In k-space this means that (34) must hold with $\hat{\phi}(k) = \hat{\phi}_c(k) + \hat{\phi}_b(k)$ and S_ϕ given by (35). Expanding the left-hand side gives

$$\begin{aligned}
\langle \hat{\phi}(k)\hat{\phi}^*(k')\rangle &= \langle [\hat{\phi}_c(k) + \hat{\phi}_b(k)][\hat{\phi}_c^*(k') + \hat{\phi}_b^*(k')]\rangle \\
&= \langle \hat{\phi}_c(k)\hat{\phi}_c^*(k')\rangle + \langle \hat{\phi}_b(k)\hat{\phi}_b^*(k')\rangle \\
&\quad + \langle \hat{\phi}_c(k)\hat{\phi}_b^*(k')\rangle + \langle \hat{\phi}_b(k)\hat{\phi}_c^*(k')\rangle \\
&= 2\pi\,[S_\phi^{(c)}(k) + S_\phi^{(b)}(k)]\,\delta(k-k') \\
&\quad + \langle \hat{\phi}_c(k)\hat{\phi}_b^*(k')\rangle + \langle \hat{\phi}_b(k)\hat{\phi}_c^*(k')\rangle
\end{aligned} \tag{36}$$

Thus we see that for the effects of each component of the spectrum to be treated separately and then superimposed requires that the cross-correlation of the phase shifts generated by each component of the spectrum vanishes. This means that we must assume much more than simply that the spectrum can be decomposed into the sum of two (or more) components as in (32) or (35) (as any spectrum obviously can be); we must furthermore assume that *each component of the spectrum represents an independent random process*. Fortunately, for the physical situation we are addressing, each component does represent an *independent source* of turbulence: $S_N^{(c)}$ (and hence $S_\phi^{(c)}$) represents a natural cascade of turbulent energy from long to short wavelengths, whereas $S_N^{(b)}$ (and $S_\phi^{(b)}$) describe the turbulent energy distribution among modes being driven by an instability.

3.2 Analysis of a δ-function "Bump" Component

With the assumption of the statistical independence of the "bump" component of the spectrum, we now proceed to analyze its effect on laser propagation. We begin with an example of the simplest type of "bump" in the turbulence spectrum: a δ-function "spike" at a single wavenumber k_b

$$S_N^{(b)}(k) \equiv 2\pi\,\delta N_0^2\,\frac{1}{k}\,\delta(k-k_b) \tag{37}$$

Here, as we again assume two-dimensional turbulence, $k \equiv \sqrt{k_x^2 + k_z^2}$, and the factor of $2\pi/k$ is appropriate for normalization. This can be seen by inserting (37) into (8) to obtain the corresponding spatial correlation function of the refractive index fluctuations

$$R_N^{(b)}(s) = \delta N_0^2\,J_0(k_b s) \tag{38}$$

with the level of the fluctuations given by $\langle \delta N^2 \rangle = \delta N_0^2$ (using (14)).

In order to construct the phase spectrum induced by this turbulent spectrum we first use the *exact* relation (12) to find

$$S_\phi^{(b)}(k_x) = k_0^2 z^2 \int \frac{dk_z}{2\pi}\, 2\pi\, \delta N_0^2 \frac{1}{k}\, \delta(k - k_b) \left(\frac{\sin k_z z/2}{k_z z/2}\right)^2 \tag{39}$$

The integral is trivially computed to give

$$S_\phi^{(b)}(k_x) = \frac{2k_0^2 z^2 \delta N_0^2}{\sqrt{k_b^2 - k_x^2}} \left(\frac{\sin\sqrt{k_b^2 - k_x^2}\, z/2}{\sqrt{k_b^2 - k_x^2}\, z/2}\right)^2 \tag{40}$$

A plot of $S_\phi^{(b)}$ is shown in Fig.3, where the parameters used are $k_0 = 6.28\mathrm{cm}^{-1}$ (carrier wavelength of 1cm), $z = 10$m (propagation distance of 10m), and $k_b = 6.28\mathrm{m}^{-1}$ (fluctuations at 1m wavelength). Although these parameter values are not of interest for laser propagation over long distances, the plot serves to illustrate the general features of this phase spectrum. It is obvious that, as one may have expected, the phase fluctuations are greatest for wavenumbers near, but below k_b, the wavenumber of the turbulent fluctuations. For $k_x = k_b$, the phase spectrum is singular (as is $S_N^{(b)}(k = k_b)$), while for $k_x > k_b$ there are no phase fluctuations at all.

Although $S_\phi^{(b)}$ is non-negative for all k_x, it oscillates with increasing frequency as k_x approaches k_b from below. The zeroes k_n of $S_\phi^{(b)}(k_x)$ are easily determined to be

$$k_n^2 = k_b^2 - n^2\left(\frac{2\pi}{z}\right)^2 \qquad 0 \le n \le \frac{k_b z}{2\pi} \tag{41}$$

which have a separation of approximately $\Delta k_n \approx 4n\pi^2/(k_b z^2)$. Thus, as $k_b z$ becomes large, the oscillations in $S_\phi^{(b)}$ become more rapid. This has the following consequence: given k_b and z, the value of $S_\phi^{(b)}$ at $0 \le k_x < k_b$ can be computed from (40). However, for large $k_b z$ (as is our usual case of interest), the value of $S_\phi^{(b)}$ at a fixed k_x is very sensitive to the value of z. Indeed, changing z by an amount of roughly $2\pi/k_b$ can change the value of $S_\phi^{(b)}$ from a maximum of $8k_0^2\delta N_0^2/(k_b^2 - k_x^2)^{3/2}$ down to zero. In practice, this means that changing a propagation distance of tens to thousands of kilometers by as little as a meter could change the phase fluctuation spectrum at any given wavenumber $k_x < k_b$.

The presence of these rapid oscillations in $S_\phi^{(b)}$, together with the fact that the phase fluctuations become singular at $k_x = k_b$ suggest that the only important fluctuations are those at $k_x = k_b$. Thus, it appears we are justified in using the approximation (13) for the phase spectrum induced by the δ-function "bump" or

$$S_\phi^{(b)}(k_x) = k_0^2 z\, S_N(k_x, k_z = 0) = 2\pi k_0^2 z\, \delta N_0^2 \frac{1}{|k_x|}\, \delta(|k_x| - k_b) \tag{42}$$

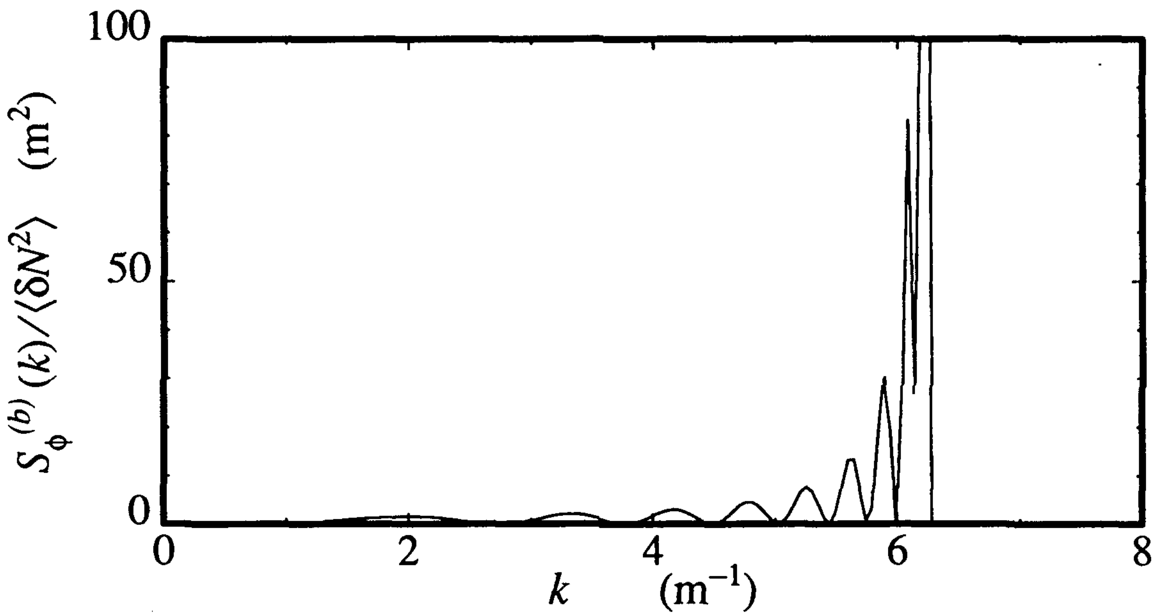

Figure 3. Plot of the exact phase spectrum (40) induced by a δ-function in the turbulence spectrum.

Even though a singular phase spectrum (such as either (40) or (42)) introduces some degree of difficulty into the numerical phase-screen method (where the value of $S_\phi^{(b)}$ is to be used as a variance for a probability distribution), the approximate expression (42) does permit theoretical analysis. Furthermore, we shall see in the next Section that an analysis of the effect of such a "spike", based on the use of (42), is useful in predicting the effect of a narrow but finite "bump" in the turbulence spectrum.

Taking (42) for the phase spectrum, we can trivially compute the phase correlation function to be

$$\begin{aligned} R_\phi^{(b)}(s_x) &= \int \frac{dk_x}{2\pi}\, 2\pi k_0^2 z\, \delta N_0^2 \frac{1}{|k_x|}\, \delta(|k_x| - k_b)\, e^{ik_x s_x} \\ &= \frac{2k_0^2 z}{k_b}\, \delta N_0^2 \cos k_b s_x \qquad (43) \end{aligned}$$

Thus, the level of the phase fluctuations induced by the "bump" is

$$\langle \phi_b^2 \rangle = \frac{2k_0^2 z}{k_b}\, \delta N_0^2 \qquad (44)$$

and by (16), the integral scalelength of the turbulence is

$$\ell_i = \frac{2}{k_b} = \frac{\ell_b}{\pi} \qquad (45)$$

where $\ell_b = 2\pi/k_b$ is the size of the turbulent eddies corresponding to the "bump" wavenumber k_b. As one would expect, the infinitesimally narrow phase spectrum (42) corresponds to a phase correlation function which has infinite range. A realistic "bump" phase spectrum will always have some finite width Δk and will therefore correspond to a correlation function which decays over a distance proportional to $(\Delta k)^{-1}$; as long as the propagation distance z of interest is much greater than this decay length, we are justified in using the results of this analysis based on (42).

To investigate the effects of such a "bump" on the propagation of a laser beam, we consider the predictions for the ensemble-averaged wave intensity $\langle I(x,z)\rangle$ and wave spectrum $\langle I(k,z)\rangle$ given by (18) and (30) of Section 2.3, respectively. We assume that the beam has an initial Gaussian wave profile (as given by (22)) so that the amplitude correlation $M_A(s;z)$ is given by (23). For the phase correlation function (43) then, the ensemble-averaged wave intensity is

$$\begin{aligned}
\langle I_b(x,z)\rangle &= \frac{k_0}{2\pi z}\int ds\; e^{-\langle\phi_b^2\rangle}\, e^{\langle\phi_b^2\rangle\cos(k_b s/2)}\sqrt{\pi}\sigma e^{-s^2k_0^2\sigma^2(z)/4z^2}\; e^{-ik_0xs/z} \\
&= \frac{\sigma k_0}{\sqrt{\pi}z}\, e^{-\langle\phi_b^2\rangle}\int_0^\infty ds\; \left[I_0(\langle\phi_b^2\rangle) + 2\sum_{n=1}^{\infty} I_n(\langle\phi_b^2\rangle)\cos(nk_b s/2)\right] \\
&\qquad\qquad \times e^{-s^2k_0^2\sigma^2(z)/4z^2}\;\cos(k_0xs/z) \\
&= \frac{\sigma}{\sigma(z)}\, e^{-\langle\phi_b^2\rangle}\Big\{I_0(\langle\phi_b^2\rangle)\, e^{-x^2/\sigma^2(z)} \\
&\qquad + \sum_{n=1}^{\infty} I_n(\langle\phi_b^2\rangle)\left[e^{-(x-x_n)^2/\sigma^2(z)} + e^{-(x+x_n)^2/\sigma^2(z)}\right]\Big\} \qquad (46)
\end{aligned}$$

Clearly, this result reduces to the vacuum propagation result (25) in the absence of turbulence (since the Bessel functions satisfy $I_0(0) = 1$ and $I_n(0) = 0$ for $n \neq 0$). If only the leading term (proportional to I_0) were present, the mean influence of the turbulence would be just a pure damping (by a factor of $I_0(\langle\phi_b^2\rangle)\exp(-\langle\phi_b^2\rangle)$) of the vacuum propagation result (although this would not conserve the beam power). Besides damping the wave intensity near the beam axis (at $x = 0$), however, the turbulence causes the beam to spread over and above that caused by natural diffraction, and it develops intensity maxima off-axis at the transverse locations $\pm x_n$ given by

$$x_n(z) \equiv \frac{n}{2}\frac{k_b}{k_0}z \qquad (47)$$

A plot of this result is shown in Fig. 4a, where it is compared with the vacuum propagation profile. For the purpose of illustration, we have chosen the following parameters

$$\begin{aligned}
k_0 &= 6280\text{cm}^{-1} \quad (\lambda = 10\mu\text{m}) \\
\sigma &= 1.0\text{m} \;\Rightarrow\; r_0 = 0.71\text{m} \\
k_b &= 6.28\text{m}^{-1} \quad (\ell_b = 1\text{m}) \\
z &= 1000\text{km} \\
\langle\phi_b^2\rangle &= 3.14 \;\Rightarrow\; \delta N_0^2 = 2.5\times 10^{-17} \;\text{ from (44)} \qquad (48)
\end{aligned}$$

With these values, the intensity peaks (47) are seen to be spaced at five meter intervals. With the use of Bessel function summation identities, one can easily show that the integral of the intensity (46) over x is indeed equal to its value at $z = 0$.

The prediction for the ensemble-averaged wave spectrum $\langle I_b(k,z)\rangle$ produced by the turbulence (30) requires the Fourier transform (31) of the turbulence modulation M_T or

$$\hat{M}_T(k;z) = \int ds\; e^{-\langle\phi_b^2\rangle}\, e^{\langle\phi_b^2\rangle\cos k_b s}\, e^{-iks}$$

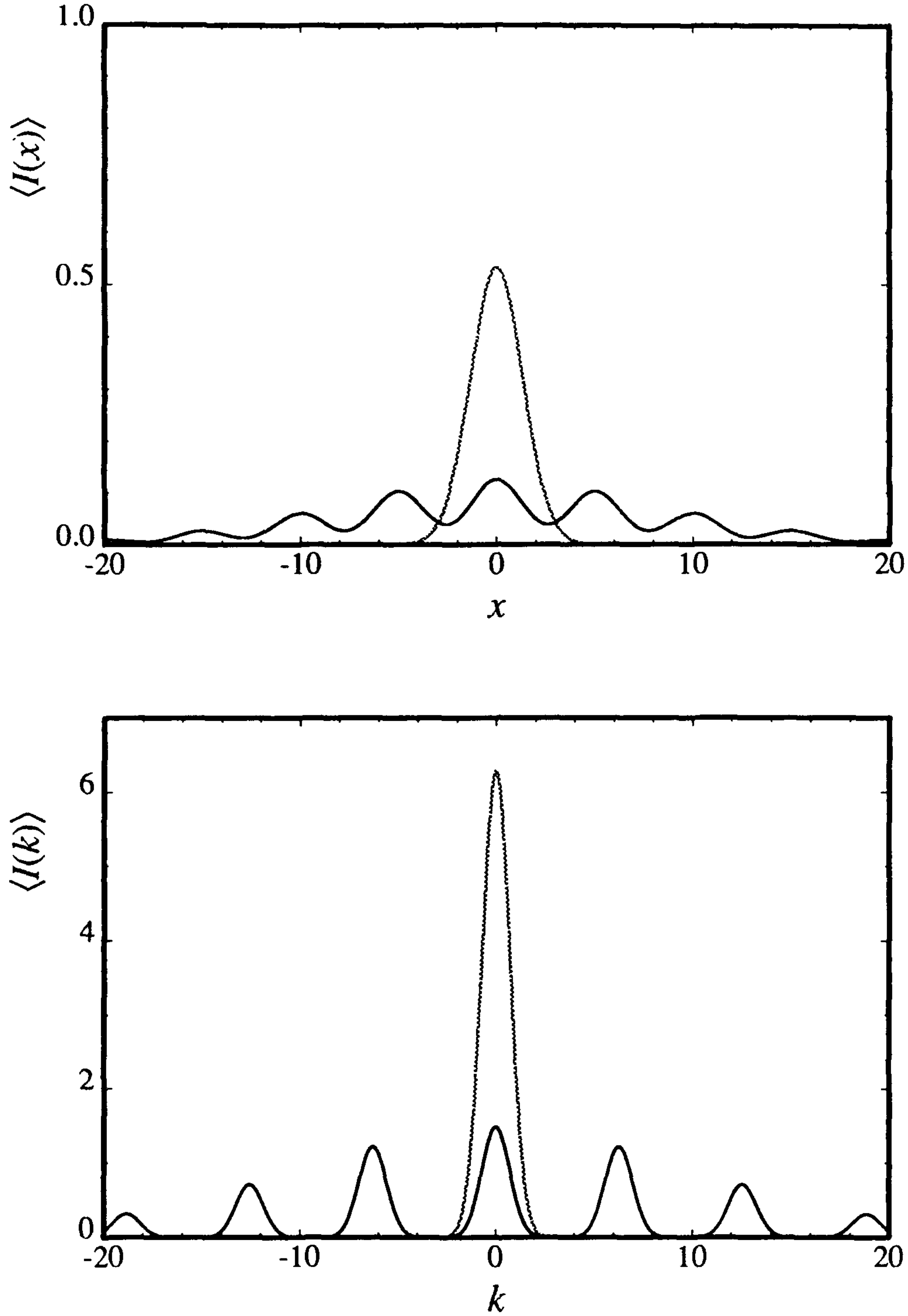

Figure 4. Plot of the predicted ensemble-averaged intensity produced by a δ-function in the turbulence spectrum, with parameters given by (48). Vacuum propagation result shown by lighter curve. (a) Spatial intensity. (b) Wave spectrum.

$$
\begin{aligned}
&= e^{-\langle\phi_b^2\rangle} \int ds\; e^{-iks} \left[I_0(\langle\phi_b^2\rangle) + 2\sum_{n=1}^{\infty} I_n(\langle\phi_b^2\rangle)\cos(nk_bs)\right] \\
&= 2\pi\, e^{-\langle\phi_b^2\rangle}\Big\{ I_0(\langle\phi_b^2\rangle)\,\delta(k) \\
&\qquad\qquad + \sum_{n=1}^{\infty} I_n(\langle\phi_b^2\rangle)\,[\delta(k-nk_b)+\delta(k+nk_b)]\Big\} \qquad (49)
\end{aligned}
$$

With the Fourier transform $\hat{\psi}(k,0)$ of the initial Gaussian profile (22)

$$
\hat{\psi}(k,0) = \sqrt{2\pi}\sigma\, e^{-k^2\sigma^2/2} \qquad (50)
$$

the form of (49) permits the trivial evaluation of (30) to give

$$
\begin{aligned}
\langle I_b(k,z)\rangle &= 2\pi\sigma^2\, e^{-\langle\phi_b^2\rangle}\Big\{ I_0(\langle\phi_b^2\rangle)\, e^{-k^2\sigma^2} \\
&\qquad + \sum_{n=1}^{\infty} I_n(\langle\phi_b^2\rangle)\left[e^{-(k-k_n)^2\sigma^2} + e^{-(k+k_n)^2\sigma^2}\right]\Big\} \qquad (51)
\end{aligned}
$$

Again, this result reduces to the vacuum expression (the square of (50)) in the absence of turbulence, and the leading term damps this value in the presence of turbulence. Like the spatial intensity (46), the wave spectrum is also spread by the turbulence, and develops sidebands at harmonics $\pm k_n$ of the turbulence "bump" wavenumber k_b:

$$
k_n = nk_b \qquad (52)
$$

A plot of this result is shown in Fig. 4b, for the same parameters (48).

Finally, one can compute from (46) the evolution of the width of the beam as it propagates As defined in (26) we have

$$
\begin{aligned}
r^2(z) &= \frac{1}{\sqrt{\pi}\sigma}\int dx\; x^2\, \langle I_b(x;z)\rangle \\
&= \frac{\sigma^2(z)}{2} + 2\, e^{-\langle\phi_b^2\rangle}\sum_{n=1}^{\infty} x_n^2 I_n(\langle\phi_b^2\rangle) \qquad (53)
\end{aligned}
$$

Here we have computed the Gaussian integrals and have again used the Bessel function summation rule. Since by (47) the intensity peaks at $\pm x_n$ are proportional to n, we use the Bessel function generating function to find

$$
\sum_{n=1}^{\infty} n^2 I_n(t) = \tfrac{1}{2}te^t \qquad (54)
$$

so that we have

$$
\begin{aligned}
r^2(z) &= \frac{\sigma^2}{2}\left(1+\frac{z^2}{k_0^2\sigma^4}\right) + 2\frac{k_b^2z^2}{4k_0^2}\frac{\langle\phi_b^2\rangle}{2} \\
&= \frac{\sigma^2}{2}\left[1+\frac{z^2}{k_0^2\sigma^4}+\frac{z^3k_b\delta N_0^2}{\sigma^2}\right] \qquad (55)
\end{aligned}
$$

where we have used the relation (44) between $\langle \phi_b^2 \rangle$ and δN_0^2. Identifying from (55) the initial width $r_0 = \sigma/\sqrt{2}$, this expression is exactly that given in (27) (indicating the cubic growth of the beam width due to turbulence) with the turbulent broadening length L_T given by

$$L_T \equiv \left[\frac{\sigma^2}{k_b \delta N_0^2} \right]^{1/3} = \left[\frac{r_0^2 \ell_b}{\pi \delta N_0^2} \right]^{1/3} \tag{56}$$

We note that this definition coincides precisely with the definition (28); it is trivial to compute from (37) that the transverse scalelength ℓ_t (defined by (29)) is exactly ℓ_b, and from (45) we have $\ell_i = \ell_b/\pi$. For the parameters (48) above, $L_T = 234$km while the Rayleigh length is $L_R = k_0 r_0^2 = 314$km; thus in this case, it is clear that the turbulent broadening dominates over natural diffraction, as borne out by Fig. 4.

3.3 Numerical Simulation for a Gaussian "Bump"

A numerical simulation of propagation through a turbulent medium characterized by the singular spectral density (37) (or corresponding singular phase spectrum (40) or (42)) using the phase-screen technique is difficult in that it utilizes fast Fourier transform methods. Therefore, we have considered a spectral density "bump" of the form

$$S_N^{(b)}(k) \equiv \frac{2\sqrt{\pi}}{\kappa k_b} \delta N_0^2 \, e^{-(k-k_b)^2/\kappa^2} \tag{57}$$

with the hope that if the width κ is small compared with the "bump" wavenumber k_b then the results of the previous Section will apply. We therefore assume the following to hold

$$\begin{aligned} k_b \equiv 2\pi/\ell_b \qquad & \kappa \equiv 2\pi/L_0 \\ k_b \gg \kappa \quad \Rightarrow & \quad L_0 \gg \ell_b \end{aligned} \tag{58}$$

where the inverse of the k-space width of the spectral density gives the range (or outer scale) L_0 of the spatial correlation function. In the numerical simulations, we will use the values

$$\begin{aligned} \ell_b &= 1\text{m} \ \ (k_b = 6.28\text{m}^{-1}) \\ L_0 &= 30\text{m} \ \ (\kappa = 0.21\text{m}^{-1}) \end{aligned} \tag{59}$$

A plot of $S_N^{(b)}$ with these parameters is shown in Fig. 5a.

In the regime (58), the two-dimensional Fourier transform (8) can be approximated analytically to give

$$R_N^{(b)}(s) \approx \delta N_0^2 \, J_0(k_b s) \tag{60}$$

which is the same as that for the δ-function "bump" (38). The exact transform (computed numerically) shown in Fig. 5c behaves like (60) for small s, but is observed

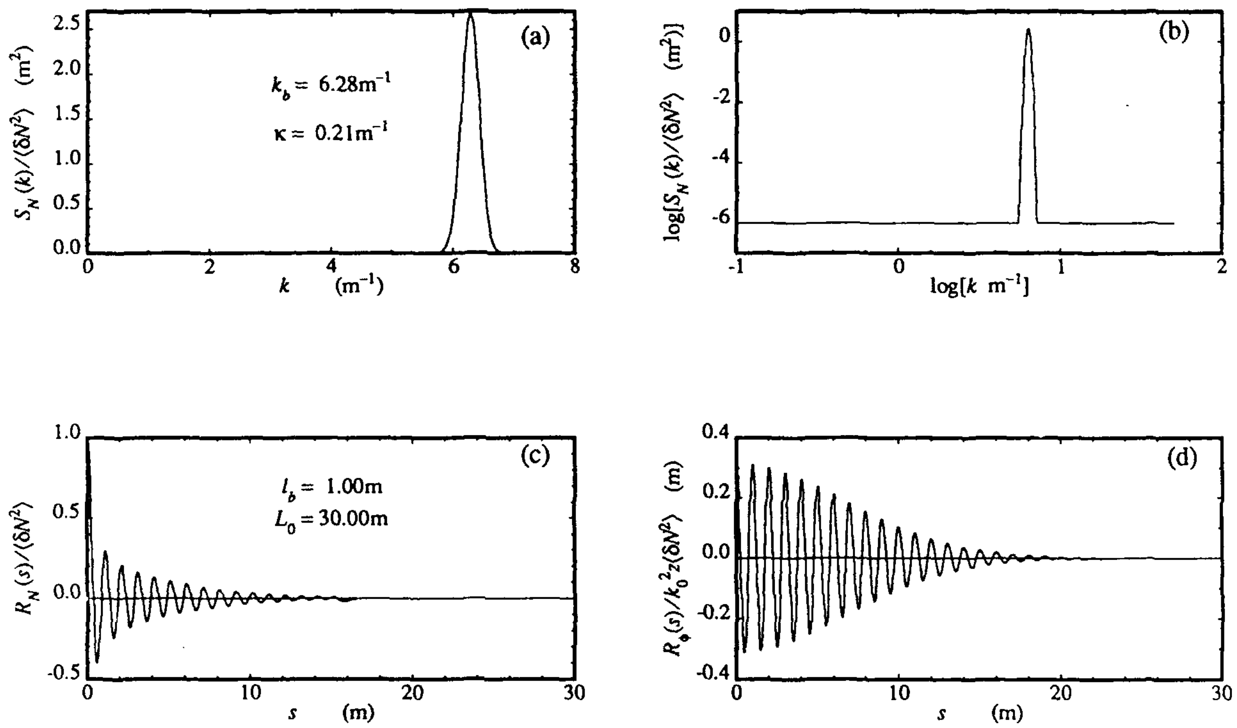

Figure 5. Turbulence functions used in numerical simulations. (a) $S_N^{(b)}(k)$ from (57) with parameters (59). (b) Log-log plot of $S_N^{(b)}(k)$. (c) Spatial correlation function $R_N^{(b)}(s)$. (d) Phase correlation function $R_\phi^{(b)}(s_x)$.

to decay much faster than J_0; indeed, the correlation function in this case should exhibit a much shorter range ($\sim L_0$) than the correlation function corresponding to a δ-function, due to the finite width of the spectral density.

As we have justified in the preceding Sections, we take the spectral density for the phase fluctuations to be approximated by (13) or

$$S_\phi^{(b)}(k_x) = k_0^2 z\, S_N(k_x, k_z = 0) = k_0^2 z \frac{2\sqrt{\pi}}{\kappa k_b} \delta N_0^2\, e^{-(|k_x| - k_b)^2/\kappa^2} \tag{61}$$

for which the phase correlation function is

$$R_\phi^{(b)}(s_x) = \frac{2k_0^2 z}{k_b}\, \delta N_0^2\, e^{-\kappa^2 s_x^2/4} \cos k_b s_x \tag{62}$$

This is plotted in Fig. 5d. Again, this form is similar to (43) for the δ-function spectral density, but now explicitly exhibits a correlation decay over the length $L_0 = 2\pi/\kappa$. The level of phase fluctuations $\langle\phi_b^2\rangle$ and the integral scalelength ℓ_i are the same as those (44,45) for the δ-function

$$\langle\phi_b^2\rangle = R_\phi^{(b)}(0) = \frac{2k_0^2 z}{k_b}\, \delta N_0^2 \;\Rightarrow\; \ell_i = \frac{2}{k_b} = \frac{\ell_b}{\pi} \tag{63}$$

while the transverse scalelength ℓ_t (from (29) using (57)) is

$$\ell_t = \frac{\sqrt{2}\ell_b L_0}{\sqrt{\ell_b^2 + 2L_0^2}} \approx \ell_b \tag{64}$$

for $L_0 \gg \ell_b$.

As in the previous Section, we would like to be able to construct the predictions (46) and (51) for the ensemble-averaged wave intensity and wave spectrum for this type of spectral "bump". Even though this is not possible analytically, we can construct these quantities numerically. Furthermore, with the non-singular spectral density (61) we can use the phase-screen method to numerically simulate the propagation of a laser beam through a realization of the turbulence described by (57). Simulating the propagation of a number of shots through an ensemble of realizations of the turbulence and averaging over the final intensities and wave spectra will allow us to check the predictions based on a numerical construction of $\langle I_b(x,z)\rangle$ and $\langle I_b(k,z)\rangle$ from (62), as well as the predictions given in the previous Section based on a δ-function spectral "bump".

Our numerical results are shown in Fig. 6. Using the parameters given in (48,59), the intensity of a beam propagated through a single realization of the turbulence is shown in Fig. 6a. The wave intensity clearly shows the effect of the loss of coherence due the turbulence; it has developed "spikes" (or "beamlets") and has spread

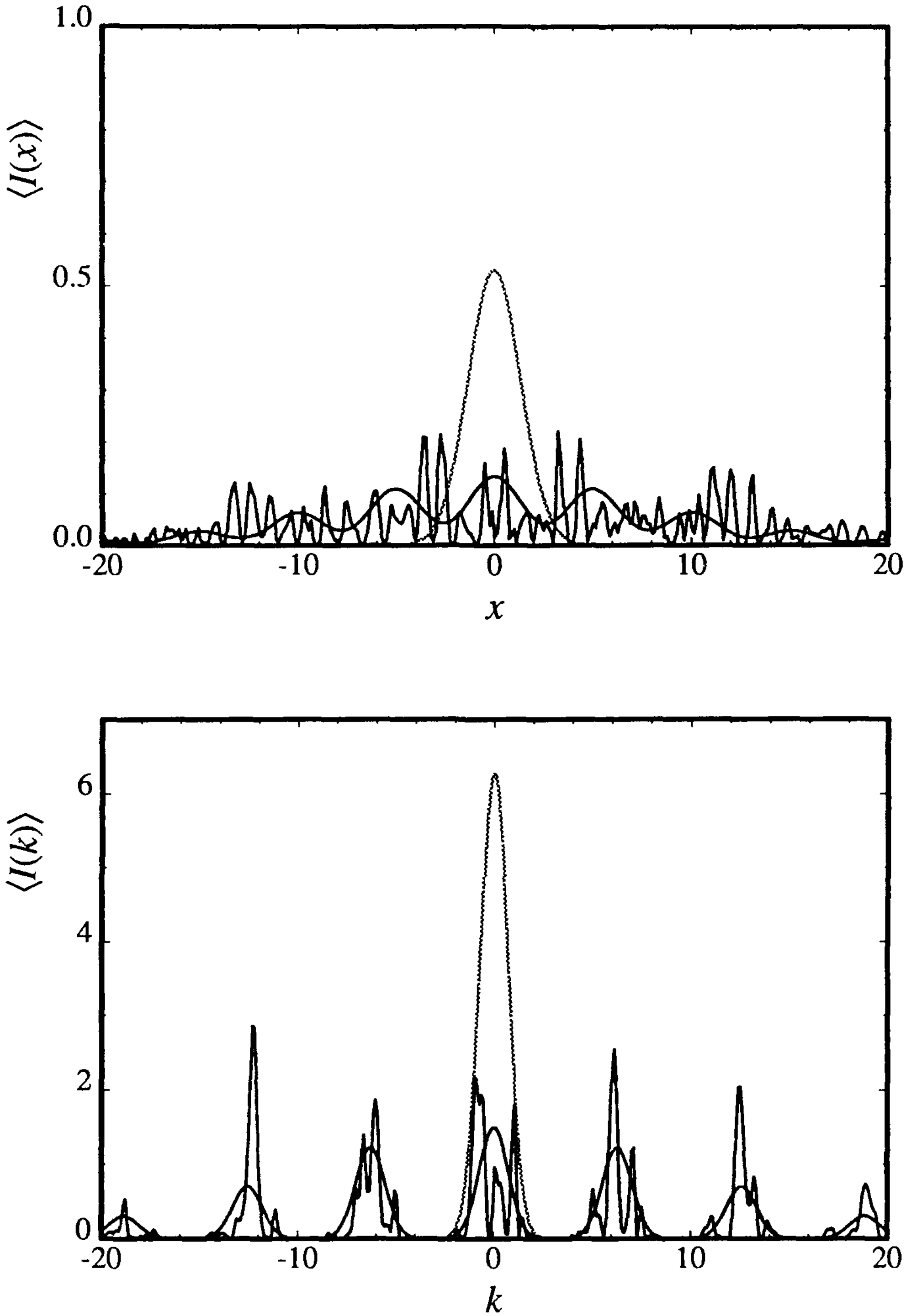

Figure 6. Numerical simulation of propagation through a single realization of Gaussian "bump" turbulence, with parameters given by (48,59), compared with prediction based on (62) (smooth dark curve) and vacuum result (light curve). (a) Intensity. (b) Wave spectrum.

much more than the vacuum result (shown by the lighter curve). The prediction for the ensemble-averaged intensity (shown by the smooth dark curve) was numerically constructed from (18) using the Gaussian "bump" phase correlation function (62); this prediction compares quite favorably with that shown in Fig. 4a based on the δ-function "bump". The wave spectrum after propagation through a single turbulence realization (shown if Fig. 6b) has clearly developed maxima at harmonics of the "bump" wavenumber k_b, and again, the ensemble-average prediction numerically constructed for the Gaussian "bump" case is very similar to the analytical form for the δ-function case shown in Fig. 4b.

An ensemble average over 100 shots (propagation through 100 realizations of the turbulence, averaged numerically) is shown in Fig. 7. Now the numerical ensemble average is much smoother than the single-realization result of Fig. 6 (the "spikes" or "beamlets" have been smoothed out by averaging, as to be expected), but the effect of the turbulence producing increased beam spreading is still evident. However, it is also interesting that the off-axis intensity maxima, as predicted by both the δ-function and Gaussian "bump" model for the ensemble-averaged intensity, do not appear in the actual numerical results. The numerical ensemble-average does spread just as predicted, but the result appears to wash out the off-axis maxima with a sort of average behavior. The wave spectrum, on the other hand, very closely matches the prediction for maxima at harmonics of the "bump" wavenumber k_b.

The discrepency between the numerical and predicted results can be explained as follows: The prediction (18) is based on a theoretical analysis of a *single phase-screen* step. That is, one can write down a closed-form expression for the final intensity (say, at $z = 1000$km as in our example) as if it were propagated in vacuum for the first half of the distance (*i.e.*, 500km), multiplied by a random phase function $\exp(i\phi(x;z))$ which accounts for the turbulence over the entire distance of propagation z, and then propagated in vacuum the remaining distance (another 500km). Another prediction could be generated by first multiplying by the random phase function (at $z = 0$) and then propagating in vacuum the entire distance ($z = 1000$km). This latter analysis would have produced an expression for the predicted ensemble-averaged intensity $\langle I(x,z)\rangle$ similar to that in (18), except for the factor of 1/2 in the argument of M_T would be unity. Using this alternative prediction in the δ-function "bump" analysis of Section 3.2 would yield off-axis intensity maxima at $x_n = nk_b z/k_0$ (*i.e.*, at $\pm 10, \pm 20\ldots$ meters for our parameters), instead of at $x_n = nk_b(z/2)/k_0$ ($\pm 5, \pm 10\ldots$ meters) as in (47). Of course, in order to conserve beam power, the intensity at each of these maxima would be lower since the profile is more spread out. With this "initial" phase-screen approach, however, the expression (30) for the wave spectrum is unchanged; thus, the wave spectrum produced by "bump" turbulence develops maxima at harmonics of the "bump" wavenumber in both analyses. The interpretation is quite clear then: a single application of the phase-screen representing a spectral

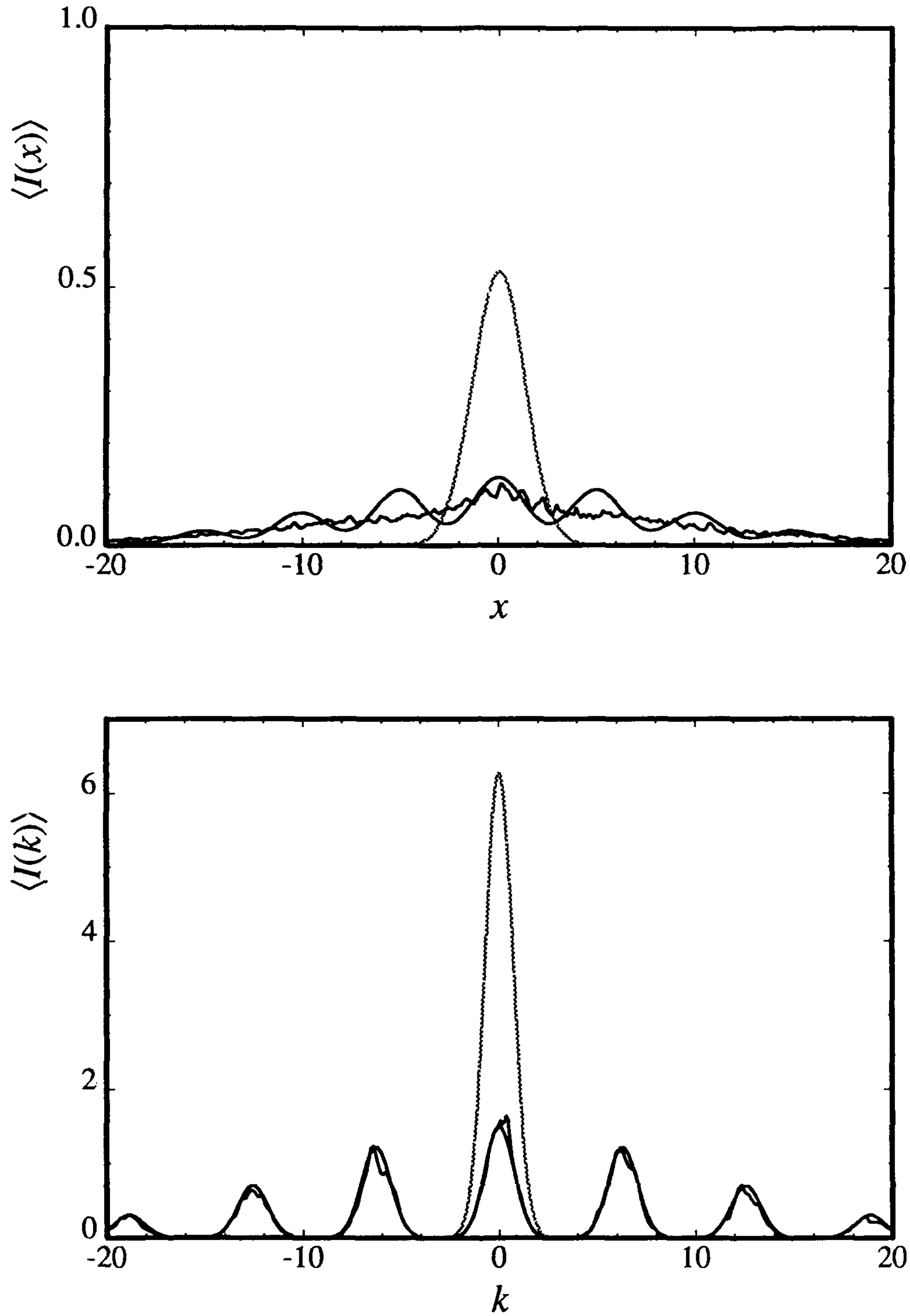

Figure 7. Numerical ensemble average of 100 shots through different realizations of Gaussian "bump" turbulence, with parameters given by (48,59), compared with prediction based on (62) (smooth dark curve) and vacuum result (light curve). (a) Intensity. (b) Wave spectrum.

bump at $k = k_b$ (irregardless of where it is placed) will cause the wave spectrum to develop peaks at harmonics $\pm k_n = \pm n k_b$. This can be thought of as a scattering of rays to a transverse wavenumber of k_n; these rays then propagate with an angle $\theta_n = k_n/k_0 = nk_b/k_0$ to the z-axis. Now, if the phase-screen is applied at the midpoint $(z/2)$, then the transverse distance reached by these rays is $x_n = \theta_n z/2 = nk_b z/(2k_0)$ (producing intensity maxima as predicted by (47)). If the phase-screen is applied at $z = 0$, however, the rays propagate twice as far at the new angles θ_n; this results in intensity maxima at the more widely spaced locations of $x_n = nk_b z/k_0$.

Naturally, in reality, propagation through turbulence is a continuous process, not accurately modeled by the imposition of a single random phase-shifting screen. Numerically, we simulate the propagation by using a series of intermediate phase-screens, with the idea that as the number of phase-screens increases (with decreasing separation), the more closely the effect of continuous propagation is simulated. Note that since by (16) the level of the phase fluctuations is proportional to the propagation distance z, it is possible to divide the total distance z up into N phase-screen intervals of length z/N (with a level of the phase fluctuations on each screen proportional to z/N) so that the total level of the phase fluctuations remains constant as N is increased. In this way, the influence of the turbulence on the beam (or the scattering of the rays) becomes more gradual. The goal in the numerical phase-screen method then is to find an optimal separation distance for the phase-screens; *i.e.*, we want to minimize the number of phase-screens required (to decrease the computation time) with the property that increasing the number does not significantly change the result. In fact, in the parameter regime we are exploring, we have found that phase-screens separated by about 100km is sufficient, with (according to (48)) a phase fluctuation level of about $\langle \phi^2 \rangle \approx 0.3$ on each phase-screen; that is the number used in the simulation results reported in this article.

So now the discrepancy between the simulation results and the predicted ensemble-averaged intensity is clear: at each of the ten phase-screens (located at z_i), rays are scattered to angles $\theta_n = nk_b/k_0$, thence to propagate the remaining distance $z - z_i$ and produce an intensity peak at $(z - z_i)\theta_n = (z - z_i)nk_b/k_0$. Since there are ten values of z_i, there are ten intensity peaks associated with each value of n: this effectively smears out the off-axis intensity maxima. If the number of phase-screens were increased (*i.e.*, more closely approximating the actual physical propagation), the initensity peaks would become more closely spaced, effectively becoming smeared so that indeed no peaks are present. Thus, it appears that using ten phase screens is enough to simulate the true continuous propagation (the smearing appears to be complete). Furthermore, we note that while the prediction for the ensemble-averaged intensity (based on the single mid-point phase-screen analysis) appears to be quite accurate (in a mean sense, averaging out the off-axis intensity maxima which are not physical); indeed, our research has shown that this prediction is much better than

one based on a single phase-screen at $z = 0$. Perhaps more importantly, even though the off-axis intensity maxima do not occur, the prediction for the growth of the beam width in terms of L_T (depending on the "bump" parameters $\ell_i = \ell_b/\pi$ and $\ell_t \approx \ell_b$) appears to be borne out by the simulations.

3.4 A "Bump" Superimposed on a Cascade

In this Section we will now consider the case of a turbulence spectral density composed of both a cascade and a "bump" component. We therefore begin with a spectral density S_N of the form

$$S_N(k) \equiv S_N^{(b)}(k) + S_N^{(c)}(k) \tag{65}$$

for which the level of refractive index fluctuations (or total power) is

$$\langle \delta N^2 \rangle \equiv \int \frac{d^2k}{(2\pi)^2}\, S_N(k) = \langle \delta N_b^2 \rangle + \langle \delta N_c^2 \rangle \tag{66}$$

Since the effects on beam propagation (such as turbulent broadening) depend on the magnitude of $\langle \delta N^2 \rangle$, it is clear from (66) that increasing the power in the "bump" component of the spectrum (*i.e.*, increasing $\langle \delta N_b^2 \rangle$) while holding the cascade component fixed will cause the beam to be more severely affected by the turbulence. For example, increasing $\langle \delta N_b^2 \rangle$ (and thus the total $\langle \delta N^2 \rangle$) will, by (16), increase the level of phase fluctuations $\langle \phi^2 \rangle$ and, by (28), decrease the turbulent broadening length L_T so that the spreading caused by the turbulence will begin to occur over a shorter distance. Furthermore, as $\langle \delta N_b^2 \rangle$ is increased to levels much higher than $\langle \delta N_c^2 \rangle$, it is reasonable to expect that the effect on the beam will be dominated by the turbulence in the "bump" component. Thus, for $\langle \delta N_b^2 \rangle \gg \langle \delta N_c^2 \rangle$, the integral and transverse scalelengths (which also affect both $\langle \phi^2 \rangle$ and L_T) will become more representative of the "bump" component; for a Gaussian "bump" as in the preceding Section, we would expect $\ell_i \to \ell_b/\pi$ and $\ell_t \to \ell_b$. In this case then, for a spectral "bump" at large k (or short wavelength) as depicted in Fig. 1, we would expect these scalelengths to be *smaller* than if the cascade component were dominant; thus, L_T is further decreased and turbulent broadening becomes a more severe effect. Even though we have seen in the previous Section that off-axis intensity maxima (as predicted using the single midpoint phase-screen theory) do not occur, the prediction for the cubic growth of the beam width in terms of L_T (depending on the "bump" parameters ℓ_i and ℓ_t) still holds.

In order to directly compare the relative effects of the cascade component $S_N^{(c)}$ and the "bump" component $S_N^{(b)}$, we can vary the magnitude of each component while holding the total power in the turbulence spectrum (or $\langle \delta N^2 \rangle$) fixed. We thus

define the fraction q of the total power in the "bump" component to be

$$q \equiv \frac{\langle \delta N_b^2 \rangle}{\langle \delta N^2 \rangle} \Rightarrow \frac{\langle \delta N_c^2 \rangle}{\langle \delta N^2 \rangle} = 1 - q \tag{67}$$

so that $\langle \delta N^2 \rangle$ by (66) is constant as q is varied. Now since (65) is a linear superposition of cascade and "bump" components, so is the phase spectrum produced by (65). Therefore, it follows that the level of phase fluctuations is

$$\begin{aligned} \langle \phi^2 \rangle &\equiv k_0^2 z \int \frac{dk}{2\pi} S_N(k) = k_0^2 z \langle \delta N^2 \rangle \ell_i \\ &= k_0^2 z [\langle \delta N_b^2 \rangle \ell_i^b + \langle \delta N_c^2 \rangle \ell_i^c] \\ &= k_0^2 z \langle \delta N^2 \rangle [q \ell_i^b + (1-q) \ell_i^c] \\ &= k_0^2 z \langle \delta N^2 \rangle [\ell_i^c - q(\ell_i^c - \ell_i^b)] \end{aligned} \tag{68}$$

As mentioned above, if the "bump" component is at larger wavenumber than the cascade (as depicted in Fig. 1), then the scalelengths characterizing the "bump" component (such as ℓ_i^b and ℓ_t^b) will be smaller than those of the cascade component. Thus, we see from (68) that as power is transferred into the bump component (q is increased), the phase fluctuations measured by $\langle \phi^2 \rangle$ are *decreased* (for $\ell_i^c > \ell_i^b$). This was previously noted by Goldring.[2]

While this result seems to indicate that taking energy out of the cascade and putting it into the "bump" tends to diminish the overall effect of the turbulence, one must remember that even though this transfer of energy decreases the magnitude of the phase fluctuations, the fluctuations are of shorter wavelength. As the scalelength of the phase fluctuations becomes smaller (especially, smaller than the beam width), the loss of transverse coherence of the beam becomes more rapid. The combination of these effects is embodied in the definition (28) of the turbulent broadening length, L_T, for which one also needs to compute the transverse scalelength ℓ_t of the turbulence. Thus, while the effective integral scalelength ℓ_i is given by (68) to be

$$\ell_i = q \ell_i^b + (1-q) \ell_i^c \tag{69}$$

the effective transverse scalelength is from (29)

$$\left(\frac{\ell_t}{2\pi} \right)^2 = \frac{4\pi^2 \langle \delta N^2 \rangle \ell_i}{\int dk\, k^2\, S_N(k)} = \frac{4\pi^2 \langle \delta N^2 \rangle \ell_i}{\int dk\, k^2\, [S_N^{(b)}(k) + S_N^{(c)}(k)]} \tag{70}$$

Using the first relation to define the transverse scalelength of each component separately this becomes

$$\frac{\ell_t^2}{\ell_i} = \langle \delta N^2 \rangle \left[\frac{\langle \delta N_b^2 \rangle \ell_i^b}{(\ell_t^b)^2} + \frac{\langle \delta N_c^2 \rangle \ell_i^c}{(\ell_t^c)^2} \right]^{-1}$$

$$= \frac{(\ell_t^c)^2}{\ell_i^c}\left[1 + q\left(\frac{(\ell_t^c)^2/\ell_i^c}{(\ell_t^b)^2/\ell_i^b} - 1\right)\right]^{-1} \tag{71}$$

Now from (28) the turbulent broadening length is

$$\begin{aligned} L_T^3(q) &= \frac{r_0^2}{\pi^2\langle\delta N^2\rangle}\frac{\ell_t^2}{\ell_i}(q) \\ &= (L_T^c)^3\left[1 + q\left(\frac{(\ell_t^c)^2/\ell_i^c}{(\ell_t^b)^2/\ell_i^b} - 1\right)\right]^{-1} \end{aligned} \tag{72}$$

where L_T^c and L_T^b are defined in terms of the total power $\langle\delta N^2\rangle$. Therefore we see that if the characteristic scalelength ℓ_t^2/ℓ_i of the cascade is greater than that for the "bump", then increasing the power in the "bump" (increasing q) will *decrease* the broadening length L_T.

We have explored this effect numerically using the phase-screen technique to propagate a beam through turbulence described by a composite spectral density such as (65). The "bump" component $S_N^{(b)}$ was taken to be the Gaussian "bump" discussed in the previous Section, while the cascade component was modeled by a von Karman spectrum

$$S_N^{(c)}(k) \equiv \frac{e^{-k^2/k_{max}^2}}{[1 + (k/k_{min})^2]^{4/3}} \tag{73}$$

In the regime $k_{min} \ll k \ll k_{max}$, this spectrum behaves like the two-dimensional Kolmogorov cascade $S_N^{(c)}(k) \sim k^{-8/3}$. The inner scale l_0^c and outer scale L_0^c of the cascade were taken to be

$$\begin{aligned} l_0^c &= 2\pi/k_{max} = 0.5\text{m} \\ L_0^c &= 2\pi/k_{min} = 10\text{m} \end{aligned} \tag{74}$$

The cascade spectrum and its corresponding correlation functions are shown in Fig. 8.

The turbulence scalelengths for this cascade were computed numerically to be

$$\begin{aligned} \ell_i^c &= 2.9\text{m} \\ \ell_t^c &= 5.1\text{m} \\ \Rightarrow\ (\ell_t^c)^2/\ell_i^c &= 9.0\text{m} \end{aligned} \tag{75}$$

For the Gaussian "bump" parameters (59), the turbulence scalelengths from (63,64) are

$$\begin{aligned} \ell_i^b &= 0.3\text{m} \\ \ell_t^b &= 1.0\text{m} \\ \Rightarrow\ (\ell_t^b)^2/\ell_i^b &= 3.3\text{m} \end{aligned} \tag{76}$$

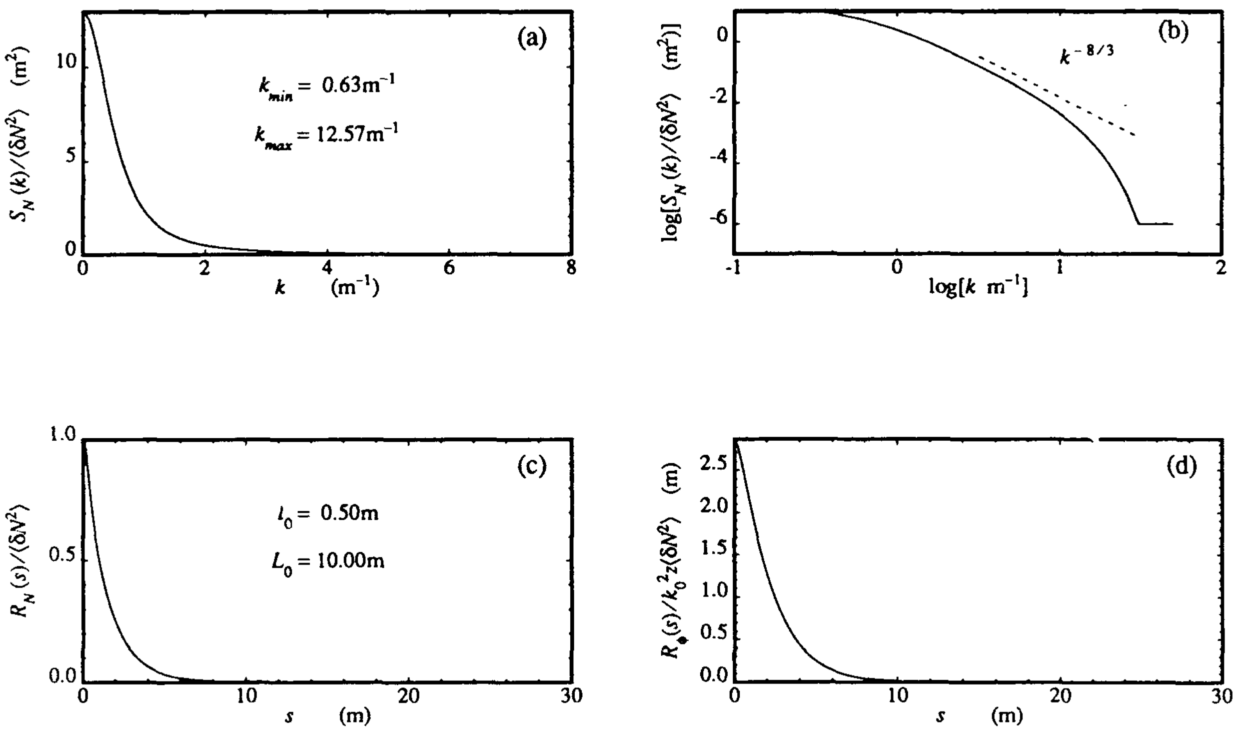

Figure 8. Cascade model used in numerical simulations. (a) $S_N^{(c)}(k)$ from (73) with parameters (74). (b) Log-log plot of $S_N^{(c)}(k)$. (c) Spatial correlation function $R_N^{(c)}(s)$. (d) Phase correlation function $R_\phi^{(c)}(s_x)$.

With these values, the turbulent broadening length for the pure cascade spectrum ($q = 0$) is $L_T^c = 330$km, while for the pure "bump" spectrum ($q = 1$) it is $L_T^b = 234$km. Thus in this case, we should observe a more pronounced broadening of the beam as energy is transferred from the cascade to the "bump".

The beam with the parameters given in (48) was propagated using the phase-screen method as described in previous Sections, with phase-screens located every 100km for a 1000km total propagation distance. We present here the results of numerically averaging over an ensemble of 100 shots through different realizations of the turbulence at fixed q. The numerical ensemble-averaged intensity and wave spectrum are compared with predictions constructed numerically, by (18) and (30), using the actual phase correlation function R_ϕ for each q.

The effect of just the pure cascade spectrum ($q = 0$) is shown in Fig. 9. Obviously, the beam has been broadened (in both x-space and k-space) due to the turbulence, but not to the degree that it was broadened by the pure "bump" turbulence ($q = 1$) as shown in Fig. 7. This is consistent with L_T for the pure cascade being larger than that for the pure "bump" spectrum. Furthermore, the wave spectrum in this case does not exhibit the harmonic structure characterizing that for the "bump" case; this is presumably because the cascade spectrum is centered at $k = 0$ and is much broader than the "bump" spectrum. Finally, it is apparent that the expressions for the ensemble-averaged intensity (18) and wave spectrum (30), based on a single midpoint phase-screen theory, give very accurate predictions for these quantities.

In Fig. 10 we show the composite spectral densities for $q = 0.325$, 0.7, and 0.83. These values were chosen because the magnitude of the "bump" component relative to that of the cascade component (*i.e.*, $S_N^{(b)}(k_b)/S_N^{(c)}(0)$) for these cases is 0.1, 0.5 and 1.0, respectively. The results of propagating through each of these turbulence models are shown in Figs. 11–13. The trend toward a broader beam as q increases is clear, as predicted above. At $q = 0.7$, the wave spectrum begins to show the harmonic structure of the pure "bump" case; the prediction for the spatial intensity also begins to develop off-axis maxima, although the numerical propagation results show that these should not be present (as explained in the preceding Section). By $q = 0.83$, the beam is broadened almost as much as in the pure "bump" case; indeed, by (72), the value of L_T is 245km for this case, compared with 234km for the pure "bump".

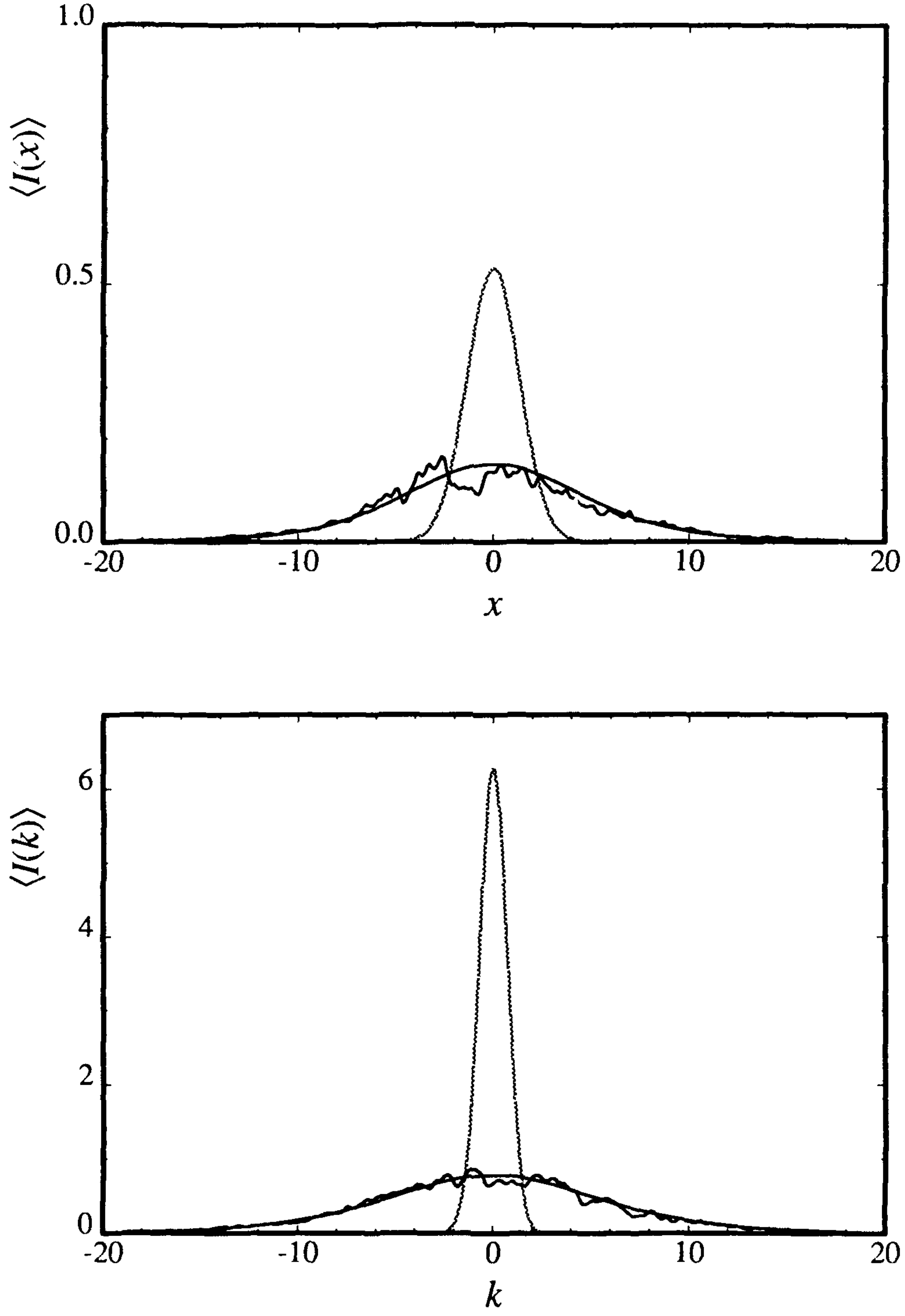

Figure 9. Numerical ensemble average of 100 shots through different realizations of cascade turbulence (73,74) and Fig. 8, compared with prediction based on (18,30) (smooth dark curve) and vacuum result (light curve). (a) Intensity. (b) Wave spectrum.

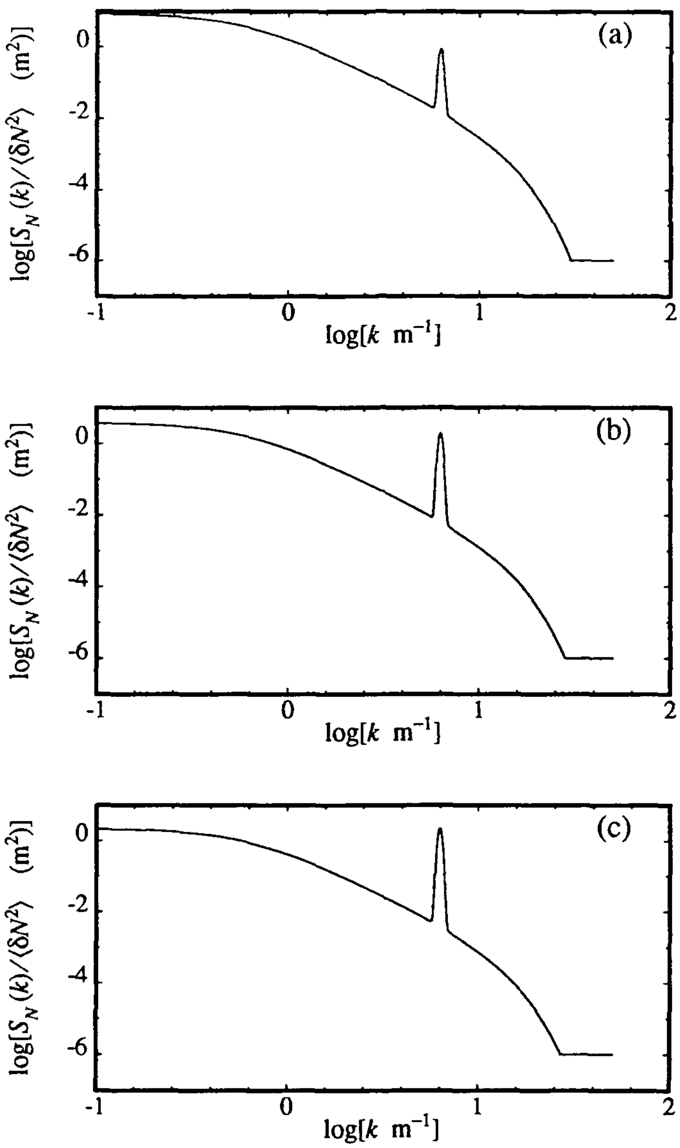

Figure 10. Composite spectral density as a function of q. (a) $q = 0.325$, relative magnitude $= 0.1$. (b) $q = 0.7$, relative magnitude $= 0.5$. (c) $q = 0.83$, relative magnitude $= 1.0$.

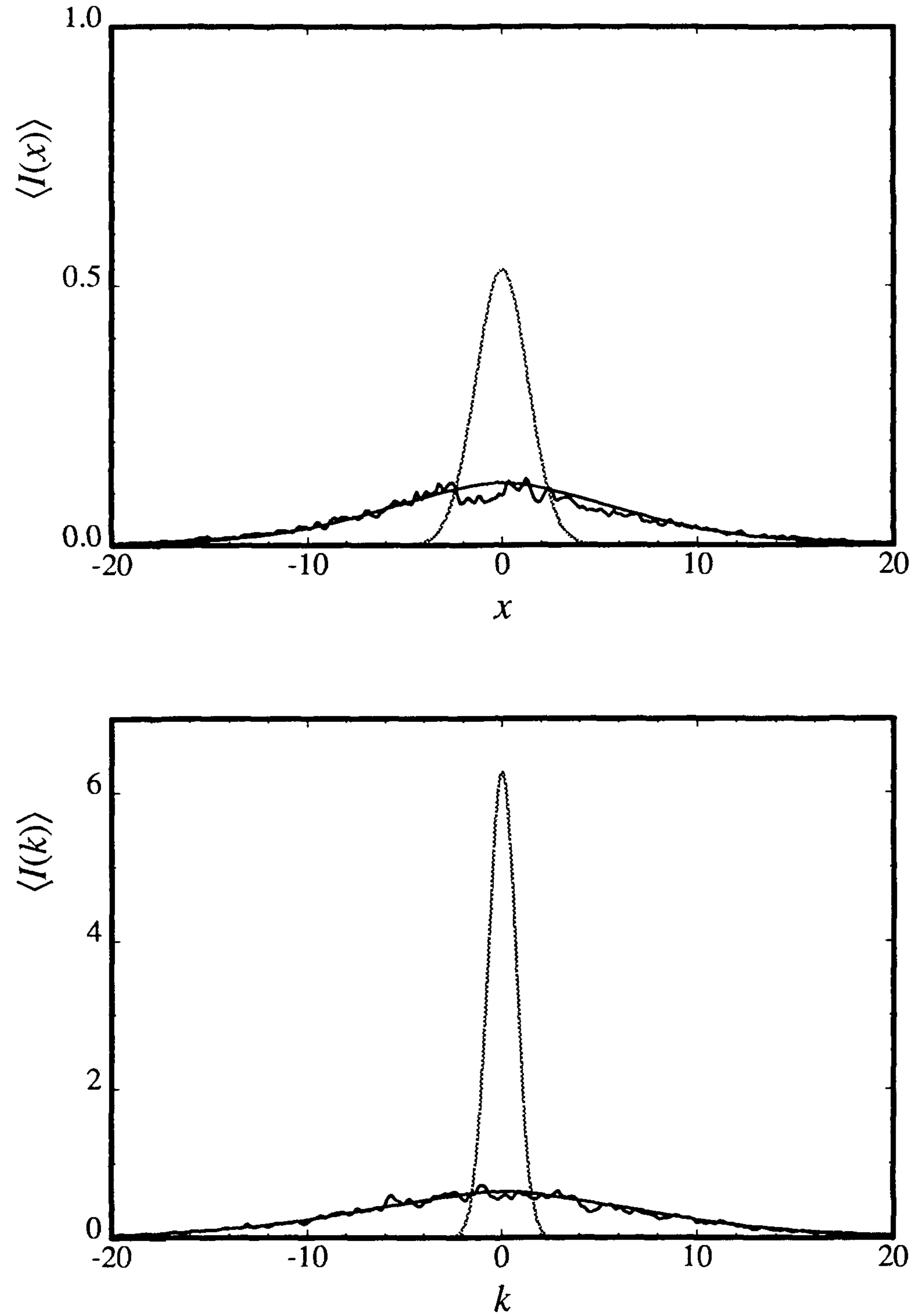

Figure 11. Numerical ensemble average of 100 shots through different realizations of turbulence with composite spectral density for $q = 0.325$. (a) Intensity. (b) Wave spectrum.

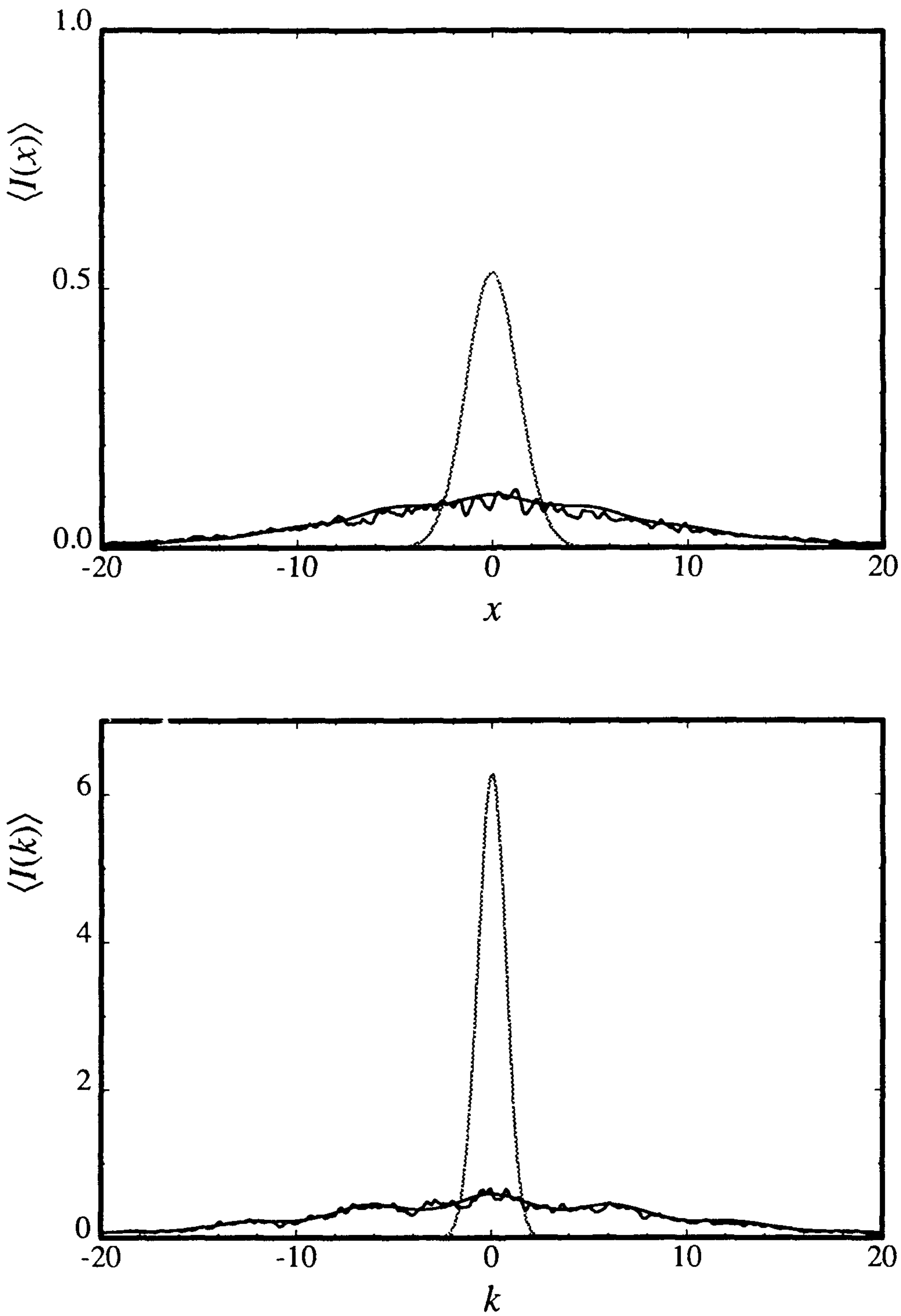

Figure 12. Numerical ensemble average of 100 shots through different realizations of turbulence with composite spectral density for $q = 0.7$.
(a) Intensity. (b) Wave spectrum.

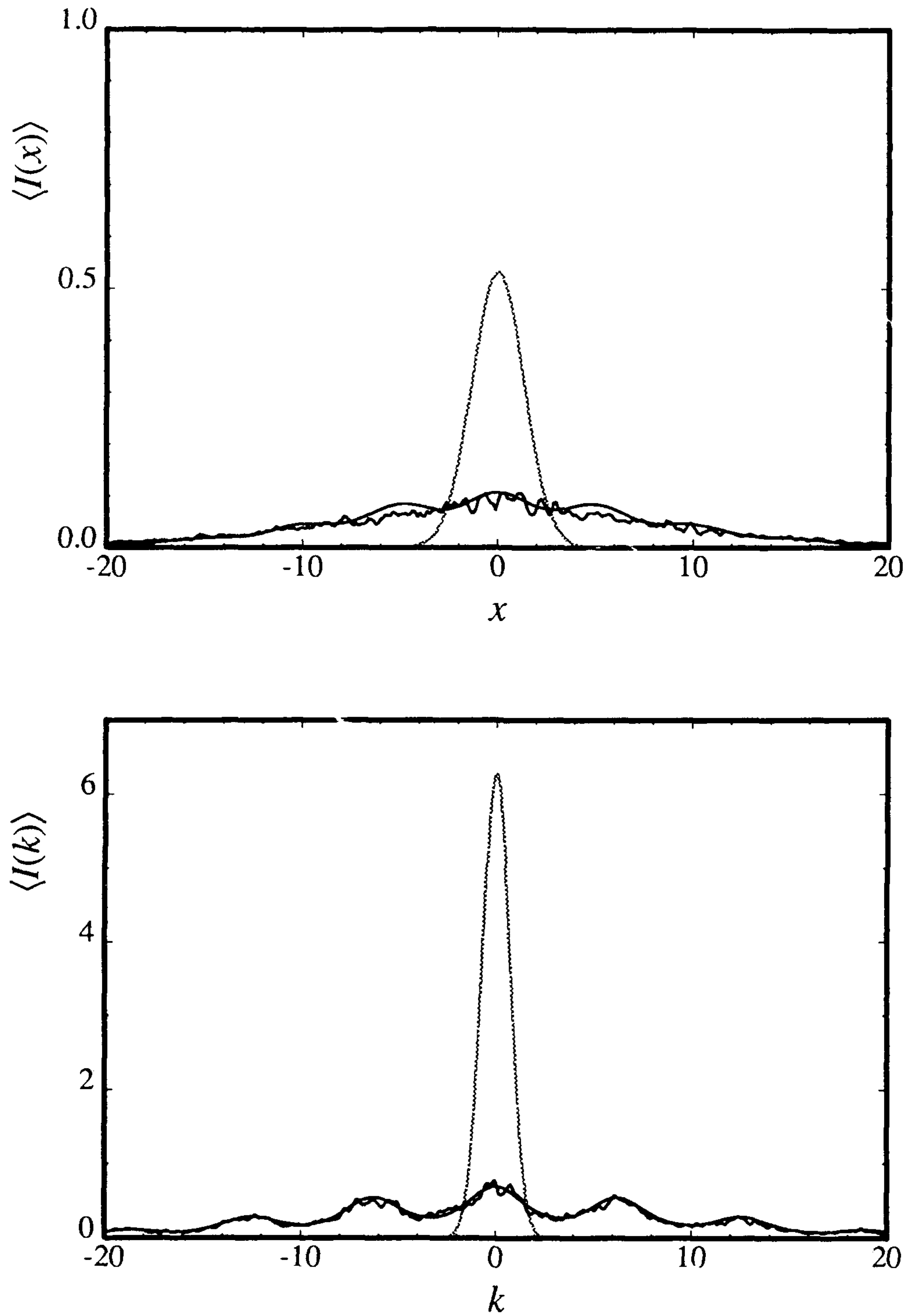

Figure 13. Numerical ensemble average of 100 shots through different realizations of turbulence with composite spectral density for $q = 0.83$. (a) Intensity. (b) Wave spectrum.

3.5 The Effect of a "Bump" in Three Dimensions

Throughout this report we have focused for simplicity on the effects of "bump" turbulence on propagation in two dimensions. Based on the analysis in the previous Section, we will now comment on the effect of a "bump" superimposed on a cascade spectrum in three dimensions. For the purpose of analytical computation, we take the isotropic cascade spectrum to be a Gaussian

$$S_N^{(c)}(k) \equiv S_c^0 \, e^{-k^2/2\kappa_c^2} \tag{77}$$

while the "bump" is also a Gaussian centered at the "bump" wavenumber k_b

$$S_N^{(b)}(k) \equiv S_b^0 \, e^{-(k-k_b)^2/2\kappa_b^2} \tag{78}$$

In these expressions, $k \gtrsim 0$ is the radius in the three-dimensional k-space. As in the previous Section, the total turbulence spectrum is simply the sum of $S_N^{(c)}$ and $S_N^{(b)}$.

The total power in each component of the spectrum can be computed to be

$$\begin{aligned} \langle \delta N_c^2 \rangle &\equiv \int \frac{d^3k}{(2\pi)^3} \, S_N^{(c)}(k) = \frac{S_c^0 \kappa_c^3}{(2\pi)^{3/2}} && (79) \\ \langle \delta N_b^2 \rangle &\equiv \int \frac{d^3k}{(2\pi)^3} \, S_N^{(b)}(k) \approx 2 \frac{S_b^0 \kappa_b k_b^2}{(2\pi)^{3/2}} && (80) \end{aligned}$$

which is the three-dimensional analogy of (66). Here we have also made the assumption that the "bump" component is very narrow and located at large wavenumber, $k_b \gg \kappa_b$ (just as in (58)). At this point it is interesting to compare the relative power in the two components:

$$p \equiv \frac{\langle \delta N_b^2 \rangle}{\langle \delta N_c^2 \rangle} = 2 \, \frac{S_b^0}{S_c^0} \, \frac{k_b^2}{\kappa_c^2} \, \frac{\kappa_b}{\kappa_c} \tag{81}$$

Consider the case where the "bump" component represents turbulence at a much smaller scalelength than the cascade; for example, let us take the following parameters:

$$\begin{aligned} k_b &\equiv 2\pi/\ell_b = 2\pi/(1\text{m}) = 6.28\text{m}^{-1} \\ \kappa_b &\equiv 2\pi/L_b = 2\pi/(10\text{m}) = 0.63\text{m}^{-1} && (82) \\ \kappa_c &\equiv 2\pi/L_c = 2\pi/(10\text{km}) = 6.28 \times 10^{-4}\text{m}^{-1} \end{aligned}$$

Thus the turbulence can be thought of as turbulent eddies in the range of $\ell_b = 1$m to $L_b = 10$m (produced by some instability, for example) superimposed on a cascade of turbulence with scalelength of $L_c = 10$km (produced by some other large-scale mechanism). In this case, the relative power (81) becomes

$$p = 2 \times 10^{11} \, \frac{S_b^0}{S_c^0} \tag{83}$$

Thus, even if the magnitude $S_b^0 = S_N^{(b)}(k_b)$ of the "bump" component is a factor of 10^{10} less than that of the cascade $S_c^0 = S_N^{(c)}(0)$, the "bump" component still has 20 times the power of the cascade component. An example of this spectrum, with the parameters as in (82), is shown in Fig. 14. In this case, the fraction q of the total power $\langle \delta N^2 \rangle = \langle \delta N_b^2 \rangle + \langle \delta N_c^2 \rangle$ that resides in the "bump" component is

$$q = \frac{p}{1+p} = 0.95 \tag{84}$$

This is a purely geometric effect of the three-dimensional **k**-space: while the "bump" is narrow and of much smaller magnitude, it is located at large radius in **k**-space and thus the k^2 volume factor works to increase the power in the "bump" component.

The effect of the "bump" component can now be assessed in terms of the turbulent broadening length L_T, using (72). For this we need the characteristic scalelengths

$$\begin{aligned} \ell_i^c &= L_c/\sqrt{2\pi} = 4.0 \times 10^3\text{m} & \ell_i^b &= \ell_b/2 = 0.5\text{m} \\ \ell_t^c &= L_c/\sqrt{2} = 7.0 \times 10^3\text{m} & \ell_t^b &= \ell_b = 1.0\text{m} \\ \Rightarrow\ (\ell_t^c)^2/\ell_i^c &= L_c\sqrt{\pi/2} = 1.25 \times 10^4\text{m} & \Rightarrow\ (\ell_t^b)^2/\ell_i^b &= 2\ell_b = 2.0\text{m} \end{aligned} \tag{85}$$

which are computed from the spectral densities using the three-dimensional versions of (16) and (29). These values give

$$\frac{(\ell_t^c)^2/\ell_i^c}{(\ell_t^b)^2/\ell_i^b} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,\frac{L_c}{\ell_b} = 6.3 \times 10^3 \tag{86}$$

Together with the value of $q = 0.95$ for this example, we see that the ratio of the relevant scalelengths is such that the turbulent broadening length (72) is decreased by a factor of almost twenty, from the value L_T^c characteristic of the cascade component to the value L_T^b determined by the "bump" component. Thus, in this case where the "bump" component *appears* to be an insignificant contribution to the total spectrum (as in Fig. 14), not only does almost all of the power reside in the "bump", but it completely determines the effects of turbulence on the beam; the "large" cascade component can be ignored.

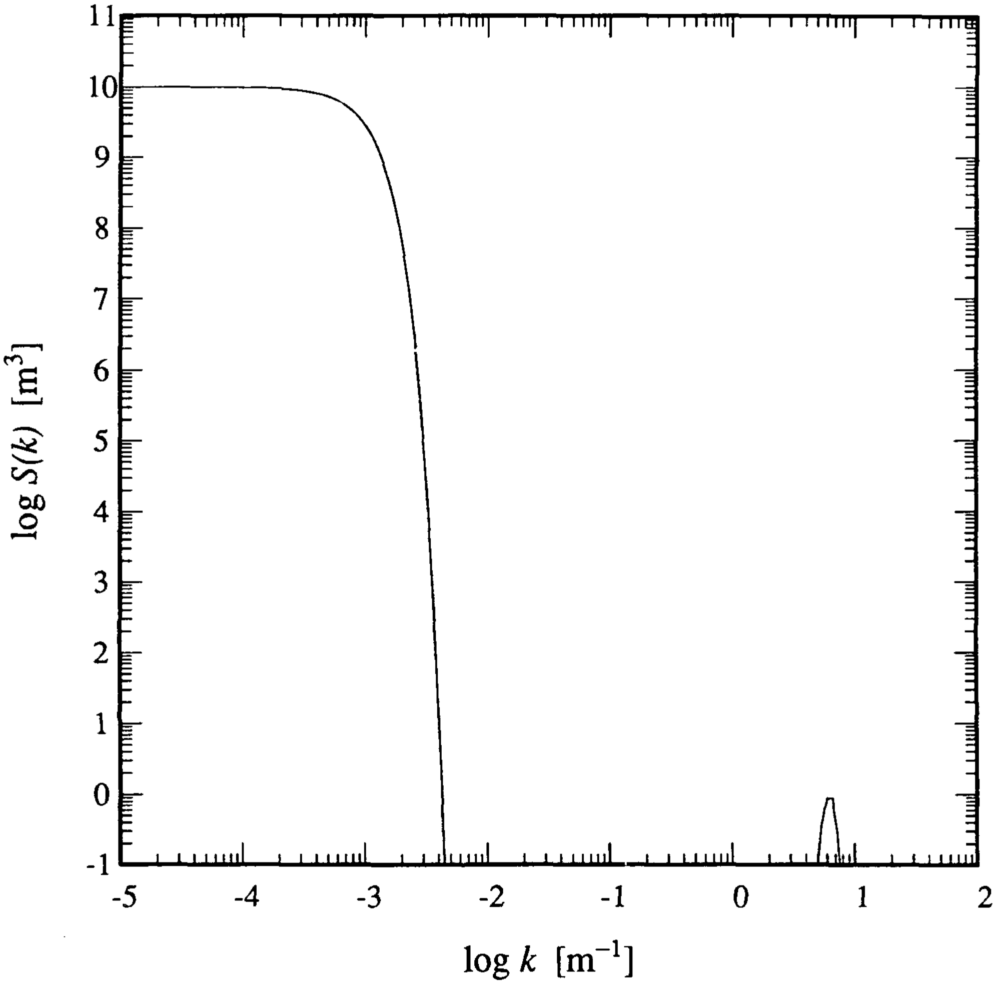

Figure 14. A three-dimensional isotropic spectrum ($k = |\mathbf{k}|$) with a "bump" superimposed on a cascade (given by (77) and (78)). The parameters are those in (82), with $S_c^0 = 10^{10}\mathrm{m}^3$ and $S_b^0 = 1\mathrm{m}^3$. In this case, 95% of the power in the spectrum resides in the "bump" component.

4 Conclusions

In this report we have anlayzed the effect of a "bump"-type turbulence spectrum on laser propagation, and we have compared our results with those produced by the standard assumption of a cascade-type spectrum. In general, the effect of narrow "bump" in the turbulence spectrum (at a finite wave number k_b) is to generate side-bands in the wave spectrum at multiples $\pm nk_b$ of the "bump" wavenumber. On average, the spatial broadening of the beam behaves as predicted (by a theory based on a modified Huygens principle) and spreads with the cube of the propagation distance; the characteristic broadening distance L_T for this effect decreases as the scalelength of the turbulent eddies $\ell_b = 2\pi/k_b$ decreases and as the level of the refractive index fluctuations $\langle \delta N_b^2 \rangle$ increases. On this basis we can conclude that given a cascade spectrum and a "bump" spectrum with the same level of fluctuations $\langle \delta N^2 \rangle$, the spectrum with the smaller characteristic scalelength will have the greater effect on the beam (because L_T for that spectrum will be smaller, and hence turbulence effects will be observed over shorter propagation distances). Since by definition a cascade spectrum is dominated by a peak at $k = 0$ and usually decreases rapidly as some power law at large k (representing a cascade of energy from longer to shorter wavelengths), it is likely that the characteristic scalelength of such a spectrum will usually be larger than that of a "bump" spectrum (which can be at any large wavenumber $k_b = 2\pi/\ell_b$); thus, for turbulent spectra of the same strength, the "bump" spectrum would usually have a greater effect on laser propagation.

We have also studied the combined effect of a "bump" component superimposed on a cascade spectrum. One important conclusion which results from this study is that in three dimensions, even if the amplitude of the "bump" component of the spectrum is much smaller than the amplitude of the cascade component (for example, by ten orders of magnitude), if the scalelength of the turbulence represented by the "bump" is much smaller than the scalelength of the cascade turbulence (say four orders of magnitude smaller), then *almost all* of the power in the spectrum is due to the "bump" component. Furthermore, in this case, the effects of the turbulence on the beam are in effect determined by the "bump" component, and the existence of the cascade component (which may represent turbulence at very large scalelengths) can be ignored.

References

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[2] T. Goldring, *Laser Beam Degradation by a Tenuous Ionized Plasma Containing a Highly Localized Spatial Frequency Source*, W.J. Schafer Associates Technical Report *#1221-37* (1990).